Quantum Field Theory II FS24

Series 6 - Renormalization of spin-$\frac{1}{2}$ theories 10.04.2024

Part 1: The LSZ formula for fermions.

At first, let's set the scene: We will first consider the case of a Dirac field Ψ and assume that its interactions respect the $U(1)$ symmetry that gives rise to the conserved current $j_\mu = \bar{\Psi}\gamma_\mu\Psi$ and its associated charge Q. In the free theory, we can create the one-particle states by acting on the vacuum state with the creation operator:

$$|p, s, +\rangle = b_s^\dagger(\mathbf{p})\,|0\rangle\,, \qquad |p, s, -\rangle = d_s^\dagger(\mathbf{p})\,|0\rangle\,, \tag{1}$$

where the label $\pm$ on the ket indicates the value of the $U(1)$ charge Q and

$$b_s^\dagger(\mathbf{p}) = \int d^3\mathbf{x}\; e^{ip\cdot x}\tilde{\Psi}(x)\gamma^0 u_s(\mathbf{p}), \qquad d_s^\dagger(\mathbf{p}) = \int d^3\mathbf{x}\; e^{ip\cdot x}\bar{v}_s(\mathbf{p})\gamma^0\Psi(x), \tag{2}$$

$$\langle p, s, +|\,\bar{\Psi}(x)\,|0\rangle = \bar{u}_s(p)e^{-ip\cdot x}, \qquad \langle p, s, -|\,\Psi(x)\,|0\rangle = v_s(p)e^{-ip\cdot x}. \tag{3}$$

Let's focus on the + charge and consider an operator that creates a particle with definite spin, localized in monentun space near $\mathbf{p}_1$, and localized in position space near the origin:

$$b_1^\dagger \equiv \int d^3\mathbf{p} f_1(\mathbf{p}) b_{s_1}^\dagger(\mathbf{p}), \tag{4}$$

where $f_1(\mathbf{p}) \propto \exp[-(\mathbf{p}-\mathbf{p_1})^2/4\sigma^2]$.

Let's now consider interactions and thus introduce the temporal variable too.

1. Prove that

$$b_1^\dagger(-\infty) - b_1^\dagger(+\infty) = i\int d^3\mathbf{p}\; f_1(\mathbf{p})\int d^4x\; \bar{\Psi}(x)(i\overleftarrow{\partial\!\!\!/} + m)u_{s_1}(\mathbf{p})e^{ip\cdot x}. \tag{5}$$

Hint: Consider that $b_1^\dagger(-\infty) - b_1^\dagger(+\infty) = -\int_{-\infty}^{+\infty} dt\; \partial_0\, b_1^\dagger(t)$ *and also that* $(\not{p} + m)u_s(\mathbf{p}) = 0$.

2. What would be the value of the expression in 1. in the free theory and why ?

3. Prove that

$$b_1(+\infty) - b_1(-\infty) = i\int d^3\mathbf{p}\; f_1(\mathbf{p})\int d^4x\; e^{-ip\cdot x}\bar{u}_{s_1}(\mathbf{p})(-i\partial\!\!\!/ + m)\Psi(x). \tag{6}$$

Let's now consider a scattering experiment with two particles.

4. Calculate the transition amplitude from an initial state $|i\rangle = \lim_{t\to-\infty} b_1^\dagger(t) b_2^\dagger(t)\,|0\rangle$ to a final state $|f\rangle = \lim_{t\to+\infty} b_{1'}^\dagger(t) b_{2'}^\dagger(t)\,|0\rangle$, which is defined as

$$< f|i >= \langle 0|\, T\; b_{2'}(+\infty) b_{1'}(+\infty) b_1^\dagger(-\infty) b_2^\dagger(-\infty)\,|0\rangle\,. \tag{7}$$

Hint: Use the properties of the time-ordering T and also consider the limit $\sigma \to 0$, where $f_1(\mathbf{p}) = \delta^3(\mathbf{p}-\mathbf{p}_1)$.

Part 2: Calculation of two loop integrals which appear in the renormalization of the spin-$\frac{1}{2}$ theory discussed in the lectures.

5. In the lectures you reached the expression for the nucleon self-energy:

$$\Sigma^f(\not{p}) = -ig^2 \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx\; \frac{-\not{p}(1-x)+m}{[k^2+p^2x(1-x)-m^2x-\mu^2(1-x)+i\epsilon]^2}. \tag{8}$$

Solve this integral analytically.

Hint: Remember how you solved a similar integral in series 3 and leave the final expression as an integral over x.

6. In the lectures you reached the following expression for the meson self-energy

$$\tilde{\Pi}^f(k^2) = ig^2 \int \frac{d^4q}{(2\pi)^4} \cdot \mathrm{Tr}\Big[\frac{i(\not{k}+\not{q}+m)}{(k+q)^2-m^2+i\epsilon} \cdot i\gamma_5 \cdot \frac{i(\not{q}+m)}{q^2-m^2+i\epsilon} \cdot i\gamma_5\Big]. \tag{9}$$

Solve this integral analytically.

Hint: Use again Feynman parametrization and express the final result as an integral over one Feynman parameter.