We have seen in the path integral formalism that we can calculate Green's functions, defined as

$$G^{(n)}(x_1, \dots, x_n) = \langle 0 | T \phi(x_1) \cdots \phi(x_n) | 0 \rangle$$

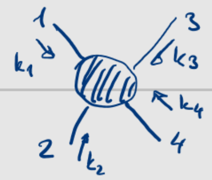
and mentioned that from these (or their Fourier transform) we can calculate S-matrix elements. The goal of these two lectures is to explain how, and to establish the link between the path integral and the canonical quant. formalism at the level of the Green's functions.

Coleman provides three different definitions of Green's functions.

1. <u>Feynman diagrams with external legs off-shell.</u>

Part of a larger diagram, so a technically defined object, a building block in the Feynm diagram approach. Clearly, one can obtain S-matrix elements if one multiplies these by the inverse of the propagators and sends the momenta of the external legs on-shell.

Example:



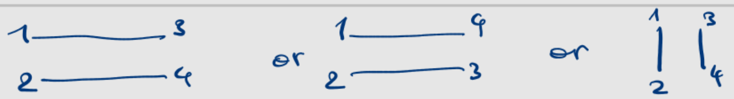
$$= \tilde{G}^{(4)}(k_1, k_2, k_3, k_4)$$

From the point of view of Feynman rules one should treat the external lines as if they were internal ones:
$$\frac{i}{k^2 - m^2 + i\varepsilon}$$
treating this diagram as if it were part of a larger one.

To obtain the S-matrix element for $2\to 2$ scattering we have to proceed as follows:

$$\langle k_3 k_4 | S-1 | k_1 k_2 \rangle = \prod_{i=1}^{4} \left(-i(k_i^2 - m^2)\right) \tilde{G}^{(4)}(k_1, k_2, -k_3, -k_4)$$

Notice that even if we include disconnected diagrams in $\tilde{G}^{(4)}$ like



they all get cancelled by the step leading to the S-matrix.

From $\tilde{G}$ we can also obtain the inverse Fourier transform

$$G^{(n)}(x_1, \dots, x_n) = \int \frac{d^4k_1}{(2\pi)^4} \cdots \frac{d^4k_n}{(2\pi)^4} \, e^{-ik_1x_1 - \dots - ik_nx_n} \, \tilde{G}^{(n)}(k_1, \dots, k_n)$$

2. Generating functional.

Take the Hamiltonian $\mathcal{H}$ and change it to

$$\mathcal{H}(x) + \rho(x)\phi(x)$$

This new vertex generates a new Feynman rule and new diagrams

$-i\tilde{\rho}(k)$ $\qquad$ $-i\tilde{\rho}(-k) = -i\tilde{\rho}^*(k)$ if $\rho(x) \in \mathbb{R}$

Define now the vacuum-to-vacuum S-matrix element in the presence of ρ as a functional $Z[\rho]$:

$$\langle 0|S|0\rangle_\rho = Z[\rho] \equiv 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int \frac{d^4k_1}{(2\pi)^4} \cdots \frac{d^4k_n}{(2\pi)^4} \, \tilde{G}^{(n)}(k_1, \dots, k_n) \tilde{\rho}(-k_1) \cdots \tilde{\rho}(-k_n)$$

By Parseval's theorem:

$$\int dx \, f(x) g(x) = \int \frac{dk}{2\pi} \tilde{f}(k) \tilde{g}^*(k) = \int \frac{dk}{2\pi} \tilde{f}(k) \tilde{g}(-k)$$

we get to:

$$Z[\rho] = 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int d^4x_1 \cdots d^4x_n \, G^{(n)}(x_1, \dots, x_n) \rho(x_1) \cdots \rho(x_n)$$

$G^{(n)}(x_1, \dots, x_n)$ are clearly Green's functions, giving the response of a system to an external perturbation $\rho(x)$.

$$G^{(n)}(x_1, \dots, x_n) = i^n \frac{\partial^n}{\partial\rho(x_1) \cdots \partial\rho(x_n)} Z[\rho]$$

$G^{(n)}$ contain all possible diagrams, including disconnected ones. If I am only interested in connected ones I can define the corresponding g.f.

$$Z_c[\rho] = 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int d^4x_1 \cdots d^4x_n \, G_c^{(n)}(x_1, \dots, x_n) \rho(x_1) \cdots \rho(x_n)$$

which turns out to be just the log of $Z[\rho]$

$$\ln Z[\rho] = Z_c[\rho] \implies Z[\rho] = \exp\left(iW[\rho]\right).$$

3. Green's functions in the Heisenberg picture.

Set $f(t) = 1$ and work with a source.

$$\mathcal{H}(x) \longrightarrow \mathcal{H}(x) + \rho(x)\phi(x)$$

Define the vacuum-to-vacuum amplitude in this framework:

$$Z[\rho] = {}_\rho\langle 0 | U_I(\infty, -\infty) | 0 \rangle_\rho$$

where U_I is the Schrödinger picture operator.

We assume that $H|0\rangle_\rho = 0$ and that ${}_\rho\langle 0|0\rangle_\rho = 1$.

$H = \int d^3x \, \mathcal{H}(x)$

Since $f(t)=1$ I cannot anymore write the v2v amplitude in terms of the S-matrix as I don't have the connection to the bare vacuum anymore. But I can still define Green's functions in this new setting.

We then have two questions:

1. Are the so defined $\tilde{G}^{(n)}(\{k_i\})$ still given by the formal sum of the Feynman diagrams?

2. Can we obtain the S-matrix elements from the Green's functions by multiplying the latter with inverse propagators and taking the on-shell limit?

The answer to the first question is positive. The answer to the second too, if we take into account a correction factor.

Green's functions in the Heisenberg picture.

Let us split the Hamiltonian into an "unperturbed" piece and the source term:

$$\mathcal{H} \to \mathcal{H} + \rho(x)\phi(x) = "\mathcal{H}_0" + "\mathcal{H}_I"$$

even though "$\mathcal{H}_0$" contains interactions and cannot be solved.

Reminder. Schrödinger picture: operators are time-indep. whereas states evolve.

$$i\frac{d}{dt}|\psi(t)\rangle_S = H|\psi(t)\rangle_S \Rightarrow |\psi(t)\rangle_S = U(t,t')|\psi(t')\rangle_S$$

$$U(t,t') = e^{-iH(t-t')} \quad \text{if } H \text{ is time-indep.}$$

Heisenberg picture: operators are time-dep. whereas states time-indep.

$$|\psi(t)\rangle_H = |\psi(0)\rangle_H = |\psi(0)\rangle_S \Rightarrow |\psi(0)\rangle_H = e^{iHt}|\psi(t)\rangle_S$$

for any operator A: $\quad A_H(t) = U^\dagger(t,0)A_S(t)U(t,0) = U(0,t)A_S(t)U^\dagger(0,t)$

Interaction picture: the states evolve only according to the interaction Hamiltonian, whereas the operators only according to the free Hamiltonian.

$$A_I(t) = U_0^\dagger(t,0)A_S(t)U(t,0) = e^{iH_0t}A_S(t)e^{-iH_0t}$$

If we set $H_I = 0$ we end up with the Heisenberg picture.

So, if we split the Hamiltonian this way we have: $\Phi_I(x)\Big|_{\rho=0} = \phi_H(x)$,

which implies:

$$Z[\rho] = {}_\rho\langle 0|U_I(\infty,-\infty)|0\rangle_\rho = {}_\rho\langle 0|T\exp\left[-i\int d^4x\, \rho(x)\phi_H(x)\right]|0\rangle_\rho$$

as Dyson's formula adapted to this case. What appears in the exponential is the Heisenberg field because operators evolve according to the free part of the Hamiltonian only, and here the free part means $\rho(x)=0$. Notice also that we cannot use Wick's theorem now because the Heisenberg fields' commutators are not c-numbers at arbitrary separations. However we can write $Z[\rho]$ as a power series:

$$Z[\rho] = 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int d^4x_1 \ldots d^4x_n \, \rho(x_1) \cdots \rho(x_n) \, {}_\rho\langle 0 | T \phi_H(x_1) \cdots \phi_H(x_n) | 0 \rangle_\rho$$

Since the two definitions of the GF $Z[\rho]$ are equivalent, we can make the following identification:

$$G^{(n)}(x_1, \ldots, x_n) = {}_\rho\langle 0 | T \phi_H(x_1) \cdots \phi_H(x_n) | 0 \rangle_\rho$$

To provide an answer to question 1 above we now need to prove that if we calculate the sum of all Feynman diagrams for the v2v amplitude in the presence of sources we get the vev of the product of n Heisenberg fields.

Let's do it:

$$Z[\rho]^{\text{Feyn}} = \lim_{\substack{t_+ \to \infty \\ t_- \to -\infty}} \frac{\langle 0 | T \exp\left[-i \int_{t_-}^{t_+} dt \int d^3x \, (\mathcal{H}_I + \rho \phi_I)\right] | 0 \rangle}{\langle 0 | T \exp\left[-i \int_{t_-}^{t_+} dt \int d^3x \, \mathcal{H}_I\right] | 0 \rangle}$$

where $|0\rangle$ is now the bare vacuum, the vacuum of H_0.

Expanding $Z[\rho]^{\text{Feyn}}$ in ρ we obtain

$$Z[\rho]^{\text{Feyn}} = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int d^4x_1 \ldots d^4x_n \, \rho(x_1) \cdots \rho(x_n) \, G^{(n)\,\text{Feyn}}(x_1, \ldots, x_n)$$

with

$$G^{(n)\,\text{Feyn}}(x_1, \ldots, x_n) = \lim_{\substack{t_+ \to \infty \\ t_- \to -\infty}} \frac{\langle 0 | T \exp\left[-i \int_{t_-}^{t_+} d^4x \, \mathcal{H}_I(x)\right] \phi_I(x_1) \cdots \phi_I(x_n) | 0 \rangle}{\langle 0 | T \exp\left[-i \int_{t_-}^{t_+} d^4x \, \mathcal{H}_I(x)\right] | 0 \rangle}$$

Effect of time-ordering on the numerator assuming $t_+ > t_1 > t_2 > \cdots > t_n > t_-$:

$$\lim_{\substack{t_+ \to \infty \\ t_- \to -\infty}} \langle 0 | T \exp\left[-i \int_{t_1}^{t_+} d^4x \, \mathcal{H}_I(x)\right] \phi_I(x_1) \cdots \phi_I(x_n) \exp\left[-i \int_{t_-}^{t_n} d^4x \, \mathcal{H}_I(x)\right] | 0 \rangle$$

$$= \lim_{\substack{t_+ \to \infty \\ t_- \to -\infty}} \langle 0 | U_I(t_+, t_1) \phi_I(x_1) U_I(t_1, t_2) \phi_I(x_2) \cdots \phi_I(x_n) U_I(t_n, t_-) | 0 \rangle$$

$$\Rightarrow G^{(n)\,\mathrm{Feyn}}(x_1,\dots,x_n) = \lim_{\substack{t_+\to\infty \\ t_-\to-\infty}} \frac{\langle 0|\, U_I(t_+,t_1)\,\phi_I(x_1)\,U_I(t_1,t_2)\,\phi_I(x_2)\cdots\phi_I(x_n)\,U_I(t_n,t_-)\,|0\rangle}{\langle 0|\,U_I(t_+,t_-)\,|0\rangle}$$

Remember now the relation between Heisenberg-picture and interaction-picture fields:

$$\phi_H(t,\vec{x}) = U_I(0,t)\,\phi_I(t,\vec{x})\,U_I(t,0)$$

$$\Rightarrow \quad U_I(t_{i-1},t_i)\,\phi_I(x_i)\,U_I(t_i,t_{i+1}) = U_I(t_{i-1},0)\underbrace{U_I(0,t_i)\,\phi_I(x_i)\,U_I(t_i,0)}_{\phi_H(x_i)}U_I(0,t_{i+1})$$

$$\Rightarrow G^{(n)\,\mathrm{Feyn}}(x_1,\dots,x_n) = \lim_{\substack{t_+\to\infty \\ t_-\to-\infty}} \frac{\langle 0|\,U_I(t_+,0)\,\phi_H(x_1)\cdots\phi_H(x_n)\,U_I(0,t_-)\,|0\rangle}{\langle 0|\,U_I(t_+,0)\,U_I(0,t_-)\,|0\rangle}$$

Let's now take the t_- limit first:

call $\langle\psi| \equiv \langle 0|\,U_I(t_+,0)\,\phi_H(x_1)\cdots\phi_H(x_n)$ and $\langle\chi| \equiv \langle 0|\,U_I(t_+,0)$

$$\lim_{t_-\to-\infty} \langle\psi|\,U_I(0,t_-)\,|0\rangle = \langle\psi|\,e^{iHt_-}e^{-iH_0t_-}\,|0\rangle = \langle\psi|\,e^{iHt_-}\,|0\rangle$$

since $H_0|0\rangle = 0$

$$H|n\rangle = E_n|n\rangle$$

$$\Rightarrow \langle\psi|\,e^{iHt_-}\,|0\rangle = \sum_n \langle\psi|n\rangle\langle n|\,e^{iHt_-}\,|0\rangle = \langle\psi|0\rangle_p\,{}_p\langle 0|\,e^{iHt_-}\,|0\rangle + \sum_{|n\rangle\neq|0\rangle_p} \langle\psi|n\rangle\langle n|\,e^{iHt_-}\,|0\rangle$$

$$= \langle\psi|0\rangle_p\,{}_p\langle 0|0\rangle + \sum_{|n\rangle\neq|0\rangle_p} e^{iE_nt_-}\langle\psi|n\rangle\langle n|0\rangle$$

$$\langle\psi|0\rangle_p\,{}_p\langle 0|0\rangle + \lim_{t_-\to-\infty}\sum_{|n\rangle\neq|0\rangle} e^{iE_nt_-}\langle\psi|n\rangle\langle n|0\rangle = \langle\psi|0\rangle_p\,{}_p\langle 0|0\rangle$$

Riemann-Lebesgue theorem

The same reasoning applied to the other limit leads to

$$G^{(n)\,Feyn}(x_1,\dots,x_n) = \lim_{\substack{t_+\to\infty \\ t_-\to-\infty}} \frac{\langle 0|U_I(t_+,0)\,\phi_H(x_1)\cdots\phi_H(x_n)\,U_I(0,t_-)|0\rangle}{\langle 0|U_I(t_+,0)\,U_I(0,t_-)|0\rangle}$$

$$= \frac{\cancel{\langle 0|0\rangle_P}\ {}_P\langle 0|\phi_H(x_1)\cdots\phi_H(x_n)|0\rangle_P\ \cancel{{}_P\langle 0|0\rangle}}{\cancel{\langle 0|0\rangle_P}\ {}_P\langle 0|0\rangle_P\ \cancel{{}_P\langle 0|0\rangle}} = G^{(n)}(x_1,\dots,x_n)$$

✓

Constructing in and out states.

We will work in the Heisenberg picture and only with physical vacuum:

$$\phi(x) \equiv \phi_H(x) \quad \text{and} \quad |0\rangle = |0\rangle_P$$

$$P^\mu|0\rangle = 0 \quad \text{and} \quad \langle 0|0\rangle = 1$$

We will consider one-meson states $|p\rangle$ normalized as

$$\langle p'|p\rangle = (2\pi)^3\, 2\omega_{\vec p}\, \delta^3(\vec p - \vec p\,')$$

$$P^\mu|p\rangle = p^\mu|p\rangle \quad , \quad \omega_{\vec p} = \sqrt{m^2 + |\vec p|^2}$$

Properties of the $\phi(x)$ we will use:

1 – $$\langle 0|\phi(x)|0\rangle = \langle 0|e^{iPx}\phi(0)e^{-iPx}|0\rangle = \langle 0|\phi(0)|0\rangle$$

$$\langle 0|\phi(0)|0\rangle = 0 \ ; \text{ if not redefine } \phi'(x) = \phi(x) - \langle 0|\phi(0)|0\rangle$$

2 – One-particle matrix elements:

$$\langle k|\phi'(x)|0\rangle = \langle k|e^{iPx}\phi'(0)e^{-iPx}|0\rangle = e^{ikx}\underbrace{\langle k|\phi'(0)|0\rangle}_{\sqrt{Z_3}}$$

3. Redefine ϕ':

$$\boxed{\phi'(x) = Z_3^{-1/2}\left(\phi(x) - \langle 0|\phi(0)|0\rangle\right) \equiv \frac{1}{\sqrt{Z_3}}\,\phi_s(x)}$$

Z_3 is called wave-function renormalization constant.

so that $$\langle k|\phi'(x)|0\rangle = \frac{1}{\sqrt{Z_3}}\langle k|\phi(x)|0\rangle = \frac{1}{\sqrt{Z_3}}\langle k|\phi(0)|0\rangle e^{ikx} = e^{ikx} \quad \boxed{\langle k|\phi'(x)|0\rangle = e^{ikx}}$$

To proceed further, we need to introduce wave packets:

$$|f\rangle = \int \frac{d^3k}{(2\pi)^3 2\omega_k} F(k)\, |k\rangle$$

with $f(x) \equiv \int \frac{d^3k}{(2\pi)^3 2\omega_k} F(k)\, e^{-ikx}$

which satisfies by construction the Klein-Gordon eq.

$$(\Box + m^2) f(x) = 0 \,.$$

Moreover we have:

$$\langle k'|f\rangle = \int \frac{d^3k}{(2\pi)^3 2\omega_k} F(k) \langle k'|k\rangle = F(k')$$

Let's now define the following operator:

$$\phi'^f(t) \equiv i \int d^3x \left[\phi'(x)\, \partial_0 f(x) - f(x)\, \partial_0 \phi'(x) \right]$$

which has zero vev by construction

$$\langle 0|\, \phi'^f(t) |0\rangle$$

Its one-particle matrix element is

$$\langle k|\phi'^f(t)|0\rangle = i \int d^3x \int \frac{d^3k'}{(2\pi)^3 2\omega_{k'}} F(k') \left[-i\omega_k' e^{-ik'x} - e^{-ik'x} \partial_0 \right] \langle k|\phi'(x)|0\rangle$$

$$= i \int d^3x \int \frac{d^3k'}{(2\pi)^3 2\omega_k'} F(k') \left(-i\omega_k' - i\omega_k \right) = F(k) = \langle k|f\rangle$$

The other one-particle matrix element is instead zero:

$$\langle 0|\phi'^f(t)|k\rangle = 0$$

In the one-particle subspace $\phi'^f(t)$ behaves like a creation operator for the normalized state $|f\rangle$.

What is the matrix element of $\phi'^f(t)$ between the vacuum and a multiparticle state $|n\rangle$?

$$\langle n|\phi'^f(t)|0\rangle = i\int d^3x \left[\partial_0 f - f\partial_0\right] e^{ip_n x}\langle n|\phi'(0)|0\rangle$$

$$= i\int \frac{d^3k}{(2\pi)^3 2\omega_k} F(\vec{k})\left(-i\omega_{\vec{k}} - iE_n\right)\langle n|\phi'(0)|0\rangle \int d^3x\, e^{i(p_n-k)x}$$

$$= \frac{\omega_{\vec{p}_n}+E_n}{2\omega_{\vec{p}_n}} \cdot F(\vec{p}_n)\langle n|\phi'(0)|0\rangle e^{-i(\omega_{\vec{p}_n}-E_n)t}$$

Because of the oscillating factor, if we send $t\to\pm\infty$ this will kill any integral over E_n.

Consider in particular the matrix element with a normalizable state $\langle\psi|$:

$$\lim_{t\to\pm\infty}\langle\psi|\phi'^f(t)|0\rangle = \lim_{t\to\pm\infty}\sum_n \langle\psi|n\rangle\langle n|\phi'^f(t)|0\rangle =$$

$$= \lim_{t\to\pm\infty}\Big[\underbrace{\langle\psi|0\rangle\langle 0|\phi'^f(t)|0\rangle}_{=0} + \sum_{|n\rangle \text{ single particle}}\langle\psi|n\rangle\langle n|\phi'^f(t)|0\rangle + \sum_{|n\rangle \text{ multi particle}}\langle\psi|n\rangle\langle n|\phi'^f(t)|0\rangle\Big]$$

$$= \int\frac{d^3k}{(2\pi)^3 2\omega_k}\langle\psi|k\rangle\underbrace{\langle k|\phi'^f(x)|0\rangle}_{F(k)} + \underbrace{\lim_{t\to\pm\infty}\sum_{|n\rangle \text{ multip.}}\langle\psi|n\rangle\left(\frac{1+E_n/\omega_{\vec{p}_n}}{2}\right)F(\vec{p}_n)e^{i(E_n-\omega_{\vec{p}_n})t}\langle n|\phi'(0)|0\rangle}_{=0 \text{ Rie-Leb.}}$$

$$= \int\frac{d^3k}{(2\pi)^3 2\omega_{\vec{k}}} F(\vec{k})\langle\psi|\vec{k}\rangle = \langle\psi|f\rangle$$

By the same calculation we would get

$$\lim_{t\to\pm\infty}\langle 0|\phi'^f(t)|\psi\rangle = 0$$

The LSZ formula.

Summary of our latest findings / definitions:

$$|f\rangle = \int \frac{d^3k}{(2\pi)^3 2\omega_{\vec{k}}} F(\vec{k})\,|k\rangle$$ is a wave packet to which we associate

$$f(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_{\vec{k}}} F(\vec{k})\,e^{-ikx}$$ which fulfills $(\Box + m^2) f(x) = 0$.

We have introduced

$$\phi'^f(t) = i \int d^3x \left[\phi'(x)\,\partial_0 f(x) - f(x)\,\partial_0 \phi'(x) \right]$$

and showed that

$$\lim_{t \to \pm\infty} \langle \psi | \phi'^f(t) | 0 \rangle = \langle \psi | f \rangle .$$

Moreover $|f\rangle \to |k\rangle$ as $f(x) \to e^{-ikx}$

How can we generalize the one-particle states $\phi'^f(t)\,|0\rangle$ which one obtains for large times to two or more particle states?

The idea is simple. We consider two functions $F_1(\vec{k})$ and $F_2(\vec{k})$ which have no overlap in their supports:

$$F_1(\vec{k})\,F_2(\vec{k}) = 0 \qquad \forall \vec{k}$$

To these we associate the functions $f_{1,2}(x)$ and the states $|f_{1,2}\rangle$.

We assert that

$$\lim_{t \to \infty} \langle \psi | \phi'^{f_2}(t) | f_1 \rangle = \langle \psi | f_1 f_2 \rangle_{out}$$

(Proof: Klaus Hepp, 1965). Also:

$$\lim_{t \to -\infty} \langle \psi | \phi'^{f_2}(t) | f_1 \rangle = \langle \psi | f_1 f_2 \rangle_{in}$$

Proof of LSZ:

$$G'^{(n)}(x_1, \dots, x_n) = \langle 0 | T\phi'(x_1) \cdots \phi'(x_n) | 0 \rangle$$

Fourier transform:

$$G'^{(n)}(x_1, \dots, x_n) = \int \frac{d^4k_1}{(2\pi)^4} \cdots \frac{d^4k_n}{(2\pi)^4} \, e^{ik_1x_1 + \dots + ik_nx_n} \, \tilde{G}'^{(n)}(k_1, \dots, k_n)$$

Let's consider the 4-point function:

$$G'^{(4)}(x_1, \dots, x_4) = \langle 0 | T\phi'(x_1) \cdots \phi'(x_4) | 0 \rangle = Z_3^{-2} G^{(4)}(x_1, \dots, x_4)$$

The formula we want to prove is:

$$\boxed{\langle k_3 k_4 | S - 1 | k_1 k_2 \rangle = \lim_{\substack{k_i^2 \to m^2 \\ i=1,\dots,4}} (-i)^4 \prod_{r=1}^{4} (k_r^2 - m^2) \tilde{G}'^{(4)}(k_1, \dots, k_4) = \lim_{\substack{k_i^2 \to m^2 \\ i=1,\dots,4}} (-i)^4 \prod_{r=1}^{4} (k_r^2 - m^2) Z_3^{-2} \tilde{G}^{(4)}(k_1, \dots, k_4)}$$

The proof will actually consider wave packets rather than plane waves:

$$\langle g_1 g_2 | S - 1 | f_1 f_2 \rangle = (i)^4 \int d^4x_1 \cdots d^4x_4 \, g_1^*(x_1) g_2^*(x_2) f_1(x_3) f_2(x_4) \times$$

$$\underbrace{\times \prod_{r=1}^{4} (\Box_r + m^2) \langle 0 | T\phi'(x_1) \cdots \phi'(x_4) | 0 \rangle}_{\text{RHS}}$$

Lemma. If $A^f(t) \equiv i \int d^3x \left[A \partial_0 f - f \partial_0 A \right]$, with A an operator, then

$$i \int d^4x \, f(x) (\Box + m^2) A(x) = \left(\lim_{t \to -\infty} - \lim_{t \to \infty} \right) A^f(t)$$

Proof.

$$i \int d^4x \, f(x) (\partial_0^2 - \vec{\nabla}^2 + m^2) A = i \int d^4x \left(f \partial_0^2 A + A(-\vec{\nabla}^2 + m^2) f \right) = i \int d^4x \left(f \partial_0^2 A - A \partial_0^2 f \right)$$

$$= i \int d^4x \, \partial_0 \left(f \partial_0 A - A \partial_0 f \right) = i \left(\lim_{t \to -\infty} - \lim_{t \to +\infty} \right) A^f(t) \quad \checkmark$$

If A is a hermitian operator, $A = A^\dagger$ then the same holds for

$$A^{f^*}(t) = -i \int d^3x \left[A \partial_0 f^* - f^* \partial_0 A \right] \quad , \quad \text{namely}$$

$$i \int d^4x \, f^*(x) \left(\Box + m^2 \right) A(x) = i \left(\lim_{t \to \infty} - \lim_{t \to -\infty} \right) A^{f^*}(t) .$$

We now use the Lemma to make the first step:

$$RHS = \left(\lim_{t_4 \to \infty} - \lim_{t_4 \to -\infty} \right) (i)^3 \int d^4x_1 \ldots d^4x_3 \; g_1^*(x_1) g_2^*(x_2) f_1(x_3) \prod_{r=1}^{3} (\Box_r + m^2) \langle 0 | T \phi'^{f_2}(t_4) \phi'(x_1) \ldots \phi'(x_3) | 0 \rangle$$

for t_{1-3} fixed and $t_4 \to \pm\infty$ there is no contact term arising from the time derivative which from outside is moved under the T-operator.

By repeating the same operation we end up with

$$RHS = \prod_{r=1}^{2} \left(\lim_{t_r \to \infty} - \lim_{t_r \to -\infty} \right) \prod_{s=3}^{4} \left(\lim_{t_s \to -\infty} - \lim_{t_s \to \infty} \right) \langle 0 | T \phi'^{g_1^*}(t_1) \phi'^{g_2^*}(t_2) \phi'^{f_1}(t_3) \phi'^{f_2}(t_4) | 0 \rangle$$

the order is important

Let's take the t_4 limits first.

$$\lim_{t_4 \to -\infty} \langle 0 | T \phi'^{g_1^*}(t_1) \phi'^{g_2^*}(t_2) \phi'^{f_1}(t_3) \phi'^{f_2}(t_4) | 0 \rangle =$$

$$\langle 0 | T \left(\phi'^{g_2^*}(t_2) \phi'^{f_1}(t_3) \phi'^{f_2}(t_4) \right) \phi'^{f_2}(-\infty) | 0 \rangle$$

$$= \langle 0 | T \left(\phi'^{g_2^*}(t_2) \phi'^{f_1}(t_3) \phi'^{f_2}(t_4) \right) | f_2 \rangle$$

whereas

$$\lim_{t_4 \to \infty} \langle 0 | T \phi'^{g_1^*}(t_1) \phi'^{g_2^*}(t_2) \phi'^{f_1}(t_3) \phi'^{f_2}(t_4) | 0 \rangle =$$

$$\langle 0 | \phi'^{f_2}(\infty) \, T \left(\phi'^{g_2^*}(t_2) \phi'^{f_1}(t_3) \phi'^{f_2}(t_4) \right) | 0 \rangle = 0$$

Now we take the t_3 limits: by the same steps we end up with

$$RHS = \prod_{r=1}^{2} \left(\lim_{t_r \to \infty} - \lim_{t_r \to -\infty} \right) \langle 0 | T \left(\phi'^{g_1^*}(t_1) \phi'^{g_2^*}(t_2) \right. | f_1, f_2 \rangle_{in}$$

The t_2 limits are less trivial, and we end up with:

$$\langle g_2|\phi'^{g_1\dagger}(t_1)|f_1f_2\rangle_{in} - \lim_{t_2\to-\infty}\langle 0|\phi'^{g_1\dagger}(t_1)\underbrace{\phi'^{g_2\dagger}(t_2)|f_1f_2\rangle_{in}}_{\substack{\shortparallel \\ |\psi\rangle\neq 0}}$$

Finally the t_1 limits give:

$$\lim_{t\to\pm\infty}\langle 0|\phi'^{f\dagger}(t)|\psi\rangle = \langle f|\psi\rangle$$

$$RHS = \left(\lim_{t_1\to\infty} - \lim_{t_1\to-\infty}\right)\left(\langle g_2|\phi'^{g_1\dagger}(t_1)|f_1f_2\rangle_{in} - \overbrace{\langle 0|\phi'^{g_1\dagger}(t_1)|\psi\rangle}\right)$$

$t_1\to\infty$ $\quad$ $t_1\to\infty$ $\quad$ $t_1\to-\infty$ $\quad$ $t_1\to-\infty$

$$= {}_{out}\langle g_1g_2|f_1f_2\rangle_{in} - \cancel{\langle g_1|\psi\rangle} - {}_{in}\langle g_1g_2|f_1f_2\rangle_{in} + \cancel{\langle g_1|\psi\rangle}$$

$$= \langle g_1g_2|S|f_1f_2\rangle - \langle g_1g_2|f_1f_2\rangle = \langle g_1g_2|S-1|f_1f_2\rangle \qquad \checkmark$$