Organization.

- Plan of the lectures, goals of this course.
- Introduce Nikos; exercises about once per week but irregular.
- Exam in a similar format as in this semester.

Counterterms

Coleman introduces 3 models:

1. $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}\mu^2\phi^2 - g\,\rho(x)\,\phi(x)$
2. " " $\rho(\vec{x})\,\phi(x)$
3. $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}\mu^2\phi^2 + \partial_\mu\psi^*\partial^\mu\psi - m^2\psi^*\psi - g\,\phi\,\psi^*\psi$

Remember: Dyson's formula

$$U_I(\infty,-\infty) = T\exp\left[-i\int d^4x\,\mathcal{H}_I(x)\right] = S$$

$$\mathcal{H}_I^{(1)} = g\,\rho(x)\,\phi(x)$$

$$\mathcal{H}_I^{(2)} = g\,f(t)\,\rho(\vec{x})\,\phi(x)$$

$$\mathcal{H}_I^{(3)} = g\,f(t)\,\psi^*\psi\,\phi$$

} $f(t)$ adiabatic function switching off the interaction $\sim \pm T/2$, with T very large.

$f(t) = f(t,T,\Delta) \to$ Plot

Because of $f(t)$, something fundamental changes in the model over time. One can see this by looking at the vacuum state.

Vacuum state of H_0 : $|0\rangle$; $H_0|0\rangle = 0$

H : $|0\rangle_p$; $(H_0 + H_I)|0\rangle_p = E_0|0\rangle_p$

Evolution of the vacuum state over time:

$t < -(T/2+\Delta)$: $|0\rangle_{t<-(T/2+\Delta)} = U(t, -(T/2+\Delta))|0\rangle = e^{-iH_0(t-(T/2+\Delta))}|0\rangle = |0\rangle$

$t = -T/2$: $|0\rangle_{t=-T/2} = \exp\left(-i\int_{-(T/2+\Delta)}^{-T/2} dt\, [H_0 + f(t)H_I(t)]\right)|0\rangle = e^{-i\gamma_-}|0\rangle_p$

$t = T/2$: $|0\rangle_{t=T/2} = U_I(T/2, -T/2)\, e^{-i\gamma_-}|0\rangle_p = e^{-i\gamma_-}e^{-iHT}|0\rangle_p = e^{-i\gamma_-}e^{-iE_0T}|0\rangle_p$

$t > T/2+\Delta$: $|0\rangle_{t>(T/2+\Delta)} = e^{-i(\gamma_+ + \gamma_- + E_0T)}|0\rangle$ ($\gamma_+ = \gamma_-$ by time reversal)

$$\Rightarrow \quad \langle 0|U(\infty,-\infty)|0\rangle = e^{-i(\gamma_+ + \gamma_- + E_0T)},$$

but because $U_I(t,0) = e^{-iH_0t}\,U(t,0)$ and $U(\infty,-\infty) = U(\infty,0)\,U(0,-\infty)$

we end up with

$$\langle 0|U(\infty,-\infty)|0\rangle = \langle 0|U_I(\infty,-\infty)|0\rangle = \langle 0|S|0\rangle$$
$$= e^{-i(\gamma_+ + \gamma_- + E_0T)}$$

which can't possibly be right because we want an answer which is independent of T. The right answer should be:

$$\lim_{T\to\infty} \langle 0|U_I(T,-T)|0\rangle = \langle 0|S|0\rangle = 1$$

This shows the need to introduce correction terms in the Lagrangian, whose only scope is to remove this phase factor in the result of the calculation of the vacuum-to-vacuum matrix element.

These correction terms are called "counterterms":

$$H_I \to \left[g \int d^3x\, \rho(\vec{x})\, \phi(\vec{x},t) - a \right] f(t)$$

This additional contribution to H_I will generate a phase in $\langle 0|0\rangle$ which is equal to:

$$a \int dt\, f(t) = a\left(T + \mathcal{O}(\Delta)\right) = \gamma_+ + \gamma_- + E_0 T$$

a can be computed order by order in perturbation theory, as an expansion in the coupling constant g.

Notice that $$\lim_{T\to\infty} a\left(T + \mathcal{O}(\Delta)\right) = \lim_{T\to\infty} \left(\gamma_+ + \gamma_- + E_0 T\right)$$

$$\Rightarrow \quad a = E_0$$

This counterterm is equal to the energy of the free vacuum.

This model can be solved exactly and shown to be fully consistent as long as we have fixed the counterterm a as we discussed.

Coleman evaluates the S-matrix and the ground-state energy and wave-funct. in this model. We will do this as exercise.

Mass renormalization and Feynman diagrams.

Consider now model 3:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}\mu_0^2\phi^2 + \partial_\mu\psi^*\partial^\mu\psi - m_0^2\psi^*\psi - g\phi\psi^*\psi$$

notice the subscript $_0$ on the masses, which has been introduced because we now expect that the coefficient in front of the quadratic term in the free Lagrangian does not have to coincide with the true mass in the interacting theory.

$\hookrightarrow$ (= energy of the one-meson state, result of a complicated calculation).

The problem is similar to the one we have encountered above in model 2 when we looked at the vacuum-to-vacuum amplitude.
The cure is also similar: introduce counterterms.

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}\mu^2\phi^2 + \partial_\mu\psi^*\partial^\mu\psi - m^2\psi^*\psi + f(t)\left[-g\psi^*\psi\phi + a + \frac{1}{2}b\phi^2 + c\psi^*\psi\right]$$

$$\Rightarrow \qquad \mathcal{H}_I = -f(t)\left[-g\psi^*\psi\phi + a + \frac{1}{2}b\phi^2 + c\psi^*\psi\right]$$

The value of these counterterms is going to be fixed by the following conditions:

$$\langle 0|S|0\rangle = 1 \Rightarrow a$$

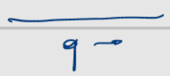
$|\vec{q}\rangle \leftrightarrow \phi$ $\qquad \langle\vec{q}|S|\vec{q}\,'\rangle = \delta^3(\vec{q}-\vec{q}\,') \Rightarrow b$

$|\vec{p}\rangle \leftrightarrow \psi$ $\qquad \langle\vec{p}|S|\vec{p}\,'\rangle = \delta^3(\vec{p}-\vec{p}\,') \Rightarrow c$

Question: having fixed the properties of single-particle states does this imply that our theory is fine also for what concerns the properties of multiparticle states? The answer will come at a later point.

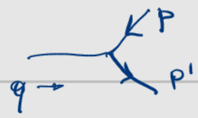
How do we do calculations in this theory?

Feynman rules:

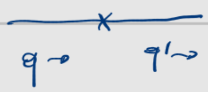


$$\int \frac{d^4q}{(2\pi)^4} \frac{i}{q^2-\mu^2+i\varepsilon}$$

$$\int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2+i\varepsilon}$$



$$-ig(2\pi)^4\,\delta^4(p+q-p')$$



$$ib(2\pi)^4\,\delta^4(q'-q)$$

$$ic(2\pi)^4\,\delta^4(p'-p)$$

$$ia(2\pi)^4\,\delta^4(0)$$

Feynman diagrams to order g^2.

$O(g)$

 and 

$= 0$ if $\mu < 2m$ $\qquad$ $= 0$ by energy-momentum conservation.

$O(g^2)$

(disconnected 2 mesons → 4 nucleons)

vacuum energy

it fixes the value of a up to $O(g^2)$ by imposing

$$\langle 0|S|0\rangle = 1$$

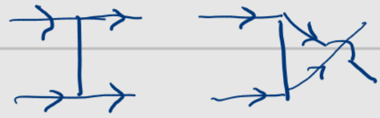
fixes the value of b up to $O(g^2)$ by the condition

$$\langle \vec{q}\,|S-1|\,\vec{q}\,'\rangle = 0$$

again this fixes the value of c up to $O(g^2)$ by the cond.

$$\langle \vec{p}\,|S-1|\,\vec{p}\,'\rangle = 0$$

We then have scattering diagrams:



etc.