Closing the circle: equivalence between the path-integral and the canonical-quantization expressions for the generating functional.

Given a Lagrangian $\mathcal{L}(\phi)$, with interaction part, we have introduced a coupling to an external source $\rho(x)$:

$$\mathcal{L}_J(\phi, J) = \mathcal{L}(\phi) + J(x)\phi(x)$$

and have shown that the generating functional:

$$Z[J] \simeq \frac{{}_B\langle 0| T e^{-i\int d^4x \left(\mathcal{H}_I(x) + J(x)\phi(x)\right)} |0\rangle_B}{{}_B\langle 0| e^{-i\int d^4x\, \mathcal{H}_I(x)} |0\rangle_B} = {}_P\langle 0| e^{-i\int d^4x\, J(x)\phi(x)} |0\rangle_P$$

can be calculated in the interaction picture with Dyson's formula, and is equivalent to the vacuum-to-vacuum amplitude in the presence of a source term.

In the path-integral approach the generating functional can be expressed as:

$$Z[J] = \frac{\int \mathcal{D}\phi\, e^{-i\int d^4x\, \mathcal{L}_J(\phi, J)}}{\int \mathcal{D}\phi\, e^{-i\int d^4x\, \mathcal{L}(\phi)}}$$

To close the circle we need to show that the two expressions of $Z[\rho]$ coincide. The only way to do this is in perturbation theory, and goes in two steps:

1. show that the expressions for $Z[J]$ for the free-field case coincide.

2. show that the perturbative expansion of $Z[J]$ obtained by pulling the interaction part of the Hamiltonian out of the matrix element or out of the path integral leads to the same expression.

Both steps are simple and are just briefly sketched here.

1. The vacuum-to-vacuum matrix element evaluated with just $J(x)\phi(x)$ as interaction Lagrangian leads to the result:

$$Z[J] = \exp\left[-\frac{1}{2}\right.$$  $$\left.\vphantom{\frac{1}{2}}\right]$$

where

$$= \int \frac{d^4k}{(2\pi)^4}\, \tilde{J}(-k)\, \frac{i}{k^2-\mu^2+i\varepsilon}\, \tilde{J}(k)$$

$$= \int d^4x\, d^4y\; J(x)\, \Delta_F(x-y)\, J(y)$$

$$\Delta_F(x-y) = \int \frac{d^4k}{(2\pi)^4}\; \frac{i}{k^2-\mu^2+i\varepsilon}\; e^{-ik(x-y)}$$

This is obtained by evaluating all possible Feynman diagrams within this theory which has only 

as interaction vertex.

We have only one connected diagram and the sum of all disconnected diagrams is given by the exponential of the connected diagrams.

The same result we have obtained with the path integral after a Wick rotation which has transformed the Minkowski integral into a Euclidean one which, for a free theory can be solved as a limiting case of a multidimensional Gaussian integral, for the number of dimensions going to infinity.

The result had the form

$$Z_E[J] = \exp\left\{ \frac{1}{2} \int d^4x_E \, J(x) \, \Delta_{F,E}(x-y) \, J(y) \right\}$$

$$\text{with} \quad \Delta_{F,E}(x) = \int \frac{d^4k_E}{(2\pi)^4} \frac{e^{-ikx}}{k^2+\mu^2}$$

which coincides with the Wick-rotated result of the calculation done with Feynman diagrams.

2. The second step is easy, as the formal manipulation to express the G.F. of an interacting theory in terms of the one for the free theory is the same for both cases:

C.Q.

$$Z[J] = N' \, {}_B\langle 0 | T[e^{-i\int d^4x \, \mathcal{H}_I}] |0\rangle_B = N' \, {}_B\langle 0 | e^{i\int d^4x \left(\mathcal{L}_I(\phi_I) + J\phi_I \right)} |0\rangle_B$$

$$= N' \exp\left[i \int d^4z \, \mathcal{L}_I\left(-i \frac{\delta}{\delta J(z)} \right) \right] \underbrace{{}_B\langle 0 | T\left[e^{i\int d^4x \, J(x)\phi_I(x)} \right] |0\rangle_B}_{Z_0[J]}$$

P.I

$$Z[J] = N\int \mathcal{D}\phi\, e^{iS[\phi,J]} = N\int \mathcal{D}\phi\, e^{iS_0[\phi,J]}\, e^{i\int d^4y\, \mathcal{L}_I(\phi)}$$

$$= N \exp\left[i\int d^4y\, \mathcal{L}_I\left(-i\frac{\delta}{\delta J(y)}\right)\right] \underbrace{\int \mathcal{D}\phi\, e^{iS_0[\phi,J]}}_{N_0^{-1} Z_0[J]}$$

$\Rightarrow N' = \frac{N}{N_0}$ proves the equivalence of the two.

Given the equivalence of the GF calculated with the PI to the one defined in CQ we can work in the PI approach and obtain S-matrix elements by applying LSZ.

As a first exercise, consider two equivalent theories:

$$\mathcal{L}_1(\phi) = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2$$

$$\mathcal{L}_2(A) = \mathcal{L}_1\left(A\left(1+\frac{1}{2}gA\right)\right) = \frac{1}{2}\partial_\mu A\,\partial^\mu A\,(1+gA)^2 - \frac{1}{2}m^2A^2\left(1+\frac{1}{2}gA\right)^2$$

In the first one the scattering amplitude $\phi\phi \to \phi\phi$ is evidently zero. In the second one, if one looks at the expression of $\mathcal{L}_2(A)$ one would at first sight guess that $AA\to AA$ is non-zero. However, an explicit calculation shows that the second amplitude is also zero, which is the conclusion one would draw by considering that $\mathcal{L}_2(A)$ is obtained from $\mathcal{L}_1(\phi)$ with a field redefinition. The latter can change the Green's functions but cannot change the S-matrix elements, because if we apply LSZ and the relation between the two fields is linear + higher powers of the field, only the linear part survives LSZ.

Counterterms and renormalization conditions.

We have seen at the beginning of this course that imposing the condition $\langle 0|0\rangle = 1$ requires the introduction of an additional term in the Lagrangian, a so-called "counterterm". This is not special for this condition, and in fact, the need for counterterms concerns other amplitudes and observables too.

We have seen, for example, that we wish to define our field so that

$$\langle 0|\phi'(0)|0\rangle = 0 \quad ; \quad \langle k|\phi'(0)|0\rangle = 1$$

I have also mentioned already that the parameter in the Lagrangian corresponding to the mass does not necessarily coincide with the pole position of the two-point function. We will now discuss all this following Coleman in Ch. 14-15.

Consider his model 3 and rewrite the Lagrangian as follows:

$$L = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}\mu^2\phi^2 + \partial_\mu\psi^*\partial^\mu\psi - m^2\psi^*\psi - g\,\psi^*\psi\phi + L_{CT}$$

where

$$\mathcal{L}_{CT} = A\phi + \frac{1}{2} B \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} C \phi^2 + D \partial_\mu \psi^* \partial^\mu \psi - E \psi^* \psi - F \psi^* \psi \phi$$

These 6 counterterms are necessary in order to impose the following conditions:

$$\langle 0|\phi|0\rangle = 0 \longleftrightarrow A$$

$$\langle k|\phi(0)|0\rangle = 1 \longleftrightarrow B$$

$$\mu = \mu_{phys} \longleftrightarrow C$$

$$\langle p|\psi(0)|0\rangle = 1 \longleftrightarrow D$$

$$m = m_{phys} \longleftrightarrow E$$

$$g = g_{phys} \longleftrightarrow F$$

The relation between conditions for the amplitudes and the relevant counterterms is quite obvious, but since we have now shown that the S-matrix and therefore all observables can be obtained from Green's functions, we should express these conditions in terms of Green's functions.

1. The condition $\langle 0|\phi|0\rangle = 0$ is already expressed in terms of a Green's function. The way to determine A goes as follows:

define $A = \sum_n A_n$ with $A_n = \mathcal{O}(g^n)$

$$\times\!\!\longrightarrow\; = \overset{(1)}{\times}\!\!\longrightarrow + \overset{(2)}{\times}\!\!\longrightarrow + \dots = \sum_n \overset{(n)}{\times}\!\!\longrightarrow$$

If one has worked out all renorm. conditions up to order g^{n-1} and can calculate the diagram

 at $O(g^n)$

we can split it into:

and by imposing that the sum is equal to zero fix A_n.

2. Källén–Lehmann representation.

Consider now the two-point function $\langle 0|T\phi(x)\phi(y)|0\rangle$, first with a fixed time-ordering: $\langle 0|\phi(x)\phi(y)|0\rangle$.

We insert a complete set of states:

$$\langle 0|\phi(x)\phi(y)|0\rangle = \sum_n \langle 0|\phi(x)|n\rangle\langle n|\phi(y)|0\rangle = \sum_n e^{-ip_n(x-y)}|\langle n|\phi(0)|0\rangle|^2$$

$$= |\langle 0|\cancel{\phi(0)}|0\rangle|^2 + \int \frac{d^3p}{(2\pi)^3 2\omega_{\vec p}} e^{-ip(x-y)}|\langle p|\phi(0)|0\rangle|^2 + \sum_n{}' e^{-ip_n(x-y)}|\langle n|\phi(0)|0\rangle|^2$$

$$= \underbrace{\int \frac{d^3p}{(2\pi)^3 2\omega_{\vec p}} e^{-ip(x-y)}}_{= \Delta_+(x-y;\mu^2)} + \underbrace{\sum_n{}' e^{-ip_n(x-y)}|\langle n|\phi(0)|0\rangle|^2}_{= \int \frac{d^4q}{(2\pi)^3}\sigma(q^2)\theta(q^0)e^{-iq(x-y)}}$$

where we have replaced

$$\sigma(q^2)\theta(q^0) = \sum_n{}'(2\pi)^3\delta^4(q-p_n)\,|\langle n|\phi(0)|0\rangle|^2$$

Where does the integral in q^2 start? If there are no bound states or meson states lighter than μ (or m), the lightest possible states are two-meson or two-nucleon states:

$$\sigma(q^2) = 0 \quad \text{if } q^2 < 4\min(\mu^2, m^2)$$

More in general we can argue that

$$\sigma(q^2) = 0 \quad \text{if } q^2 < \mu^2 + \eta$$ with $\eta > 0$ and depending on the dynamics of the theory.

Moreover, from the definition of $\sigma(q^2)$ it is clear that

$$\sigma(q^2) \geq 0 \,.$$

Rewrite

$$\langle 0 | \phi(x) \phi(y) | 0 \rangle = \Delta_+(x-y; \mu^2) + \int \frac{d^4 q}{(2\pi)^3} \int_0^\infty ds\, \delta(s - q^2)\, \sigma(s)\, \theta(q^0)\, e^{-iq(x-y)}$$

Since

$$\int_{q^0} d^4 q\, \delta(s - q^2)\, \theta(q^0) = \frac{d^3 q}{2\omega_{s,\vec{q}}}$$

we can write the second term as

$$\int_0^\infty ds\, \sigma(s)\, \Delta_+(x-y; s)$$

and then, putting everything together:

$$\langle 0 | \phi(x) \phi(y) | 0 \rangle = \int_0^\infty ds\, \rho(s)\, \Delta_+(x-y; s)$$

$$\text{with} \qquad \rho(s) = \delta(s - \mu^2) + \sigma(s)$$

The result would be the same, mutatis mutandis, with a different ordering of the arguments, which allows us to calculate the commutator, for example:

$$\langle 0 | [\phi(x), \phi(y)] | 0 \rangle = i \int_0^\infty ds\, \rho(s)\, \Delta(x-y; s)$$

If we take the equal-time commutator of the field and its time-derivative we obtain something interesting:

$$\langle 0 | [\phi(\vec{x}, t), \dot{\phi}(\vec{y}, t)] | 0 \rangle = Z_3^{-1} \langle 0 | [\phi_s(\vec{x}, t), \dot{\phi}(\vec{y}, t)] | 0 \rangle = Z_3^{-1}\, i\, \delta^3(\vec{x} - \vec{y})$$

where $\phi_s = Z_3\left(\phi - \langle 0|\phi|0\rangle\right)$ and ϕ_s satisfies canonical comm. rel. by constr.

On the other hand, since we know that

$$i\frac{\partial}{\partial x_0}\Delta(x-y;s)\Big|_{x^0=y^0} = -i\,\delta^3(\vec{x}-\vec{y})$$

we can conclude that

$$\langle 0|[\phi(\vec{x},t),\dot{\phi}(\vec{y},t)]|0\rangle = i\,\delta^3(\vec{x}-\vec{y})\left[1+\int_0^\infty ds\,\sigma(s)\right]$$

$$\Rightarrow \qquad \underbrace{1+\int_0^\infty ds\,\sigma(s) = Z_3^{-1}}_{\text{Lehmann's sum rule}} \geq 1 \quad \Rightarrow \quad Z_3 \leq 1$$