Källén-Lehmann representation

Consider now the two-point function $\langle 0|T\phi(x)\phi(y)|0\rangle$, first with a fixed time-ordering: $\langle 0|\phi(x)\phi(y)|0\rangle$.

We insert a complete set of states:

$$\langle 0|\phi(x)\phi(y)|0\rangle = \sum_n \langle 0|\phi(x)|n\rangle\langle n|\phi(y)|0\rangle = \sum_n e^{-ip_n(x-y)}|\langle n|\phi(0)|0\rangle|^2$$

$$= |\langle 0|\cancel{\phi(0)}|0\rangle|^2 + \int\frac{d^3p}{(2\pi)^3 2\omega_{\vec p}} e^{-ip(x-y)}|\langle p|\phi(0)|0\rangle|^2 + \sum_n{}' e^{-ip_n(x-y)}|\langle n|\phi(0)|0\rangle|^2$$

$$= \underbrace{\int\frac{d^3p}{(2\pi)^3 2\omega_{\vec p}} e^{-ip(x-y)}}_{=\,\Delta_+(x-y;\mu^2)} + \underbrace{\sum_n{}' e^{-ip_n(x-y)}|\langle n|\phi(0)|0\rangle|^2}_{=\,\int\frac{d^4q}{(2\pi)^3}\sigma(q^2)\theta(q^0)e^{-iq(x-y)}}$$

where we have replaced

$$\sigma(q^2)\theta(q^0) = \sum_n{}'(2\pi)^3\delta^4(q-p_n)\,|\langle n|\phi(0)|0\rangle|^2$$

Where does the integral in q^2 start? If there are no bound states or meson states lighter than μ (or m), the lightest possible states are two-meson or two-nucleon states:

$$\sigma(q^2)=0 \quad \text{if } q^2 < 4\min(\mu^2,m^2)$$

More in general we can argue that

$$\sigma(q^2)=0 \quad \text{if } q^2<\mu^2+\eta$$ with $\eta>0$ and depending on the dynamics of the theory.

Moreover, from the definition of $\sigma(q^2)$ it is clear that

$$\sigma(q^2)\geq 0.$$

Rewrite

$$\langle 0|\phi(x)\phi(y)|0\rangle = \Delta_+(x-y;\mu^2) + \int\frac{d^4q}{(2\pi)^3}\int_0^\infty ds\,\delta(s-q^2)\sigma(s)\theta(q^0)e^{-iq(x-y)}$$

Since $\int_{q^0} d^4q\, \delta(s-q^2)\,\theta(q^0) = \frac{d^3q}{2\omega_{s,\vec{q}}}$ we can write the second term as

$\int_0^\infty ds\, \sigma(s)\, \Delta_+(x-y;s)$ and then, putting everything together:

$$\langle 0|\phi(x)\phi(y)|0\rangle = \int_0^\infty ds\, \rho(s)\, \Delta_+(x-y;s)$$

with $\rho(s) = \delta(s-\mu^2) + \sigma(s)$

Källén-Lehmann or spectral representation

spectral function.

The result would be the same, mutatis mutandis, with a different ordering of the arguments, which allows us to calculate the commutator, for example:

$$\langle 0|[\phi(x),\phi(y)]|0\rangle = i\int_0^\infty ds\, \rho(s)\, \Delta(x-y;s)$$

Reminder: $i\Delta(x-y;\mu^2) = \Delta_+(x-y;\mu^2) - \Delta_+(y-x;\mu^2) = \int \frac{d^3p}{(2\pi)^3 2\omega_{\vec{p}}}\left[e^{-ip(x-y)} - e^{ip(x-y)}\right] \leftarrow \begin{cases} =0 \\ \text{for } (x-y)^2<0. \end{cases}$

If we take the equal-time commutator of the field and its time-derivative we obtain something interesting:

$$\langle 0|[\phi(\vec{x},t),\dot{\phi}(\vec{y},t)]|0\rangle = Z_3^{-1}\langle 0|[\phi_s(\vec{x},t),\dot{\phi}(\vec{y},t)]|0\rangle = Z_3^{-1}\, i\,\delta^3(\vec{x}-\vec{y})$$

where $\phi_s = Z_3\left(\phi - \langle 0|\phi|0\rangle\right)$ and ϕ_s satisfies canonical comm. rel. by constr.

On the other hand, since we know that

$$i\frac{\partial}{\partial x_0}\Delta(x-y;s)\Big|_{x^0=y^0} = -i\,\delta^3(\vec{x}-\vec{y})$$

we can conclude that

$$\langle 0|[\phi(\vec{x},t),\dot{\phi}(\vec{y},t)]|0\rangle = i\,\delta^3(\vec{x}-\vec{y})\left[1+\int_0^\infty ds\,\sigma(s)\right]$$

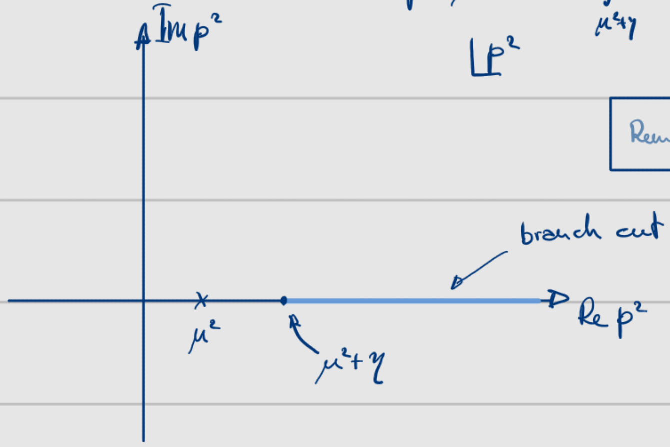
$$\Rightarrow \quad \underbrace{1+\int_0^\infty ds\,\sigma(s)}_{\text{Lehmann's sum rule}} = Z_3^{-1} \geq 1 \quad\Rightarrow\quad Z_3 \leq 1$$

Let us now include time ordering and get the Green's function

$$G^{(2)}(p,p') \equiv (2\pi)^4 \delta^4(p+p') \tilde{D}_r(p^2)$$

where

$$\tilde{D}_r(p^2) = \frac{i}{p^2-\mu^2+i\varepsilon} + \int_{\mu^2+\eta}^{\infty} ds\, \sigma(s)\, \frac{i}{p^2-s+i\varepsilon}$$



$\boxed{\text{Remember: } \sigma(q^2)\theta(q^0) = \sum_n{}' (2\pi)^3 \delta^4(q-p_n) \left|\langle n|\phi(0)|0\rangle\right|^2}$

$-i\tilde{D}_r(p^2)$ is analytic everywhere other than at μ^2 and along the branch cut.

Schwarz reflection principle: $-i\tilde{D}_r(p^2)$ is real on the real axis for $p^2<\mu^2+\eta$ and $p^2\neq\mu^2$ $\Rightarrow$ $\left[-i\tilde{D}_r(p^2)\right]^* = -i\tilde{D}_r(p^{2*})$

$\Rightarrow$ discontinuity across the branch cut $-i2\pi\sigma(s)$

after using the Sokhotski-Plemelj formula: $\lim_{\varepsilon\to 0} \frac{1}{x+i\varepsilon} = -i\pi\delta(x) + P\frac{1}{x}$.

All in all we can conclude that

$$\tilde{D}_r(p^2) = \frac{i}{p^2-\mu^2+i\varepsilon} + (\text{analytic function around } \mu^2)$$

pole of $\tilde{D}_r(p^2)$: $\quad p^2=\mu^2$ = physical mass of the meson

residue of $\tilde{D}_r(p^2)$: $\quad i \quad (\Longleftrightarrow \langle p|\phi(0)|0\rangle = 1)$

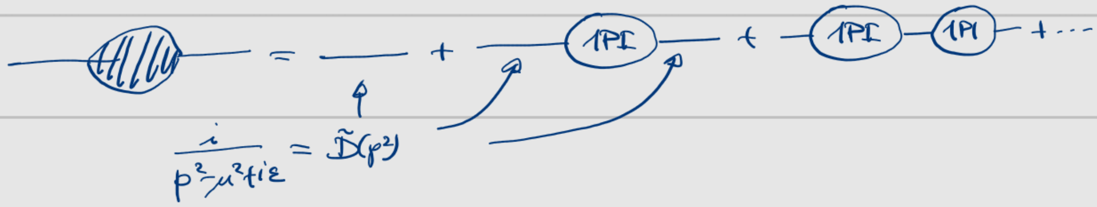
These are the conditions that we want to be satisfied in a healthy theory and which will allow us to determine the relevant counterterms.

The latter step requires a bit more work before being implemented in a practical way. In particular, it is useful to introduce the:

<u>Self-energy</u>.

1PI diagram = a diagram which cannot be separated into two pieces by cutting just a single internal line.

Define:  $\equiv -i\tilde{\Pi}_r(p^2)$

$\tilde{D}_r(p^2) \equiv$ 

$$\tilde{D}_r(p^2) = \tilde{D} + \tilde{D}\left[-i\tilde{\Pi}_r(p^2)\right]\tilde{D} + \tilde{D}\left[-i\tilde{\Pi}_r(p^2)\right]\tilde{D}\left[-i\tilde{\Pi}_r(p^2)\right]\tilde{D} + \dots$$

$$= \frac{i}{p^2-\mu^2-\tilde{\Pi}_r(p^2)+i\varepsilon} \quad , \text{ applying: } \frac{i}{x}\left(1+\frac{\Pi_r}{x}+\left(\frac{\Pi_r}{x}\right)^2+\dots\right) = \frac{i}{x-\Pi_r}$$

<u>Renormalization conditions:</u>

apply a power-series expansion of $\tilde{\Pi}_r(p^2)$ around μ^2:

$$\tilde{\Pi}_r(p^2) = \tilde{\Pi}_r(\mu^2) + (p^2-\mu^2)\tilde{\Pi}_r'(\mu^2) + \tilde{\tilde{\Pi}}_r(p^2)$$

For μ^2 to be the mass the denominator must have a zero at $p^2=\mu^2$

so $\quad \tilde{\Pi}_r(\mu^2)=0$

and for the residue to be i we must have

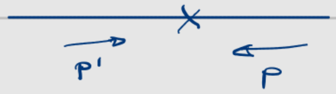
$$\tilde{\Pi}_r'(\mu^2)=0$$

These two conditions allow us to determine the values of B and C.

If we work in pert. theory, we do it order by order in the coupling constant g.

Indeed if we calculate the contribution of the CT B and C to the self-energy we get the following:

remember: $\mathcal{L}_{CT} = \dots \frac{B}{2}\partial_\mu\phi\partial^\mu\phi - \frac{C}{2}\phi^2$

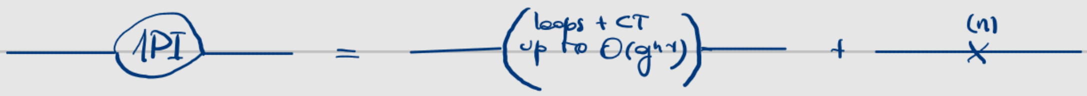


$$i(2\pi)^4\delta^4(p+p')\left[Bp^2 - C\right]$$

We then formally expand B and C in powers of g

$$B = \sum_n B_n = \sum_n b_n g^n \quad \text{and} \quad C = \sum_n C_n = \sum_n c_n g^n$$

and the program to determine the coeff. b_n and c_n works as follows:



$$= (\text{known + calculable contr.}) + i(B_n p^2 - C_n)$$

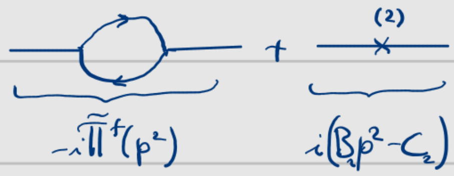
which fixes B_n and C_n in terms of calculable contributions:

$$(iB_n\mu^2 - C_n) = -(\text{known + calculable contr.})\Big|_{p^2=\mu^2}$$

$$iB_n = -\frac{d}{dp^2}(\text{known + calculable contr.})\Big|_{p^2=\mu^2}$$

To see how this program works in practice let's calculate the self-energy to $O(g^2)$.

$-i\tilde{\Pi}_r(p^2) =$



$$\tilde{\Pi}_r(\mu^2) = 0 \implies -i\tilde{\Pi}^f(\mu^2) + iB_2\mu^2 - iC_2 = 0 \implies C_2 = B_2\mu^2 - \tilde{\Pi}^f(\mu^2)$$

$$\frac{d}{dp^2}\tilde{\Pi}_r(p^2)\Big|_{p^2=\mu^2} = 0 \implies -i\frac{d}{dp^2}\tilde{\Pi}^f(p^2)\Big|_{\mu^2} + iB_2 = 0 \implies B_2 = \frac{d}{dp^2}\tilde{\Pi}^f(p^2)\Big|_{p^2=\mu^2}$$

$$\implies \cancel{i}\tilde{\Pi}_r(p^2) = \cancel{-i}\left[\tilde{\Pi}^f(p^2) - \tilde{\Pi}^f(\mu^2) - \frac{d}{dp^2}\tilde{\Pi}^f(p^2)\Big|_{\mu^2}(p^2-\mu^2)\right]$$

Let us now calculate $\tilde{\Pi}^f(p^2)$, and do this by just applying Feynman rules:

$$-i\tilde{\Pi}^f(p^2) = (-ig)^2 \int \frac{d^4q}{(2\pi)^4} \frac{i}{q^2-m^2+i\varepsilon} \frac{i}{(q+p)^2-m^2+i\varepsilon}$$

Clearly, by power-counting, the integral is divergent. We will still proceed with the calculation. As a first step we apply Feynman's trick for combining denominators:

$$\int_0^1 dx \frac{1}{[ax+b(1-x)]^2} = -\frac{1}{a-b} \frac{1}{ax+b(1-x)}\Big|_0^1 = \frac{1}{b-a}\left[\frac{1}{a}-\frac{1}{b}\right] = \frac{1}{ab}$$

which gives:

$$-i\tilde{\Pi}^f(p^2) = g^2 \int \frac{d^4q}{(2\pi)^4} \int_0^1 dx \frac{1}{\left[q^2(1-x)+(q+p)^2x-m^2+i\varepsilon\right]^2} =$$

$$= g^2 \int \frac{d^4q}{(2\pi)^4} \int_0^1 dx \frac{1}{\left[q^2+(2pq+p^2)x-m^2+i\varepsilon\right]^2}$$

$$(q+px)^2 = q^2+2pqx+p^2x^2$$

$$\Rightarrow \quad = g^2 \int \frac{d^4q}{(2\pi)^4} \int_0^1 dx \frac{1}{\left[q^2+p^2x(1-x)-m^2+i\varepsilon\right]^2}$$

If we look at the difference $\tilde{\Pi}^f(p^2)-\tilde{\Pi}^f(\mu^2)$

we immediately realize that it is finite because the integrand will go like $\frac{1}{q^6}$ for $q\to\infty$. The same is true for the derivative

$\frac{d}{dp^2}\Pi^f(p^2)\Big|_{\mu^2}$, so everything we need to calculate

$\tilde{\Pi}_r(p^2)$ is actually finite and calculable.

Regularizations.

In the discussion above we haven't emphasized the determination of the counterterms, but rather the finite result, which is finite. Counterterms, on the other hand, can be divergent, and in order to control how they diverge and how the infinities cancel out in the final result, it is necessary to introduce an additional intermediate step called regularization. One modifies by hand the divergent integral and makes it finite but dependent on an external parameter. Of course if this parameter is removed (by taking an appropriate limit) the integrals diverge again. Calculations have to be set up in such a way that the counterterms, fixed by the renormalization conditions, exactly compensate the divergent part of the loop integrals, so that if one removes the regularization parameter, the total stays finite.

There are different ways to regularize integrals and we are going to briefly introduce just two.

1. Cut-off regularization

Consider the integral

$$I_n = \int \frac{d^4q}{(2\pi)^4} \frac{1}{(q^2+a+i\varepsilon)^n}$$

After a Wick rotation: $q_0 \to i q_4$, $q^2 \to -q_E^2$ we get

$$I_n = i \int \frac{d^4 q_E}{(2\pi)^4} \frac{1}{(-q_E^2+a+i\varepsilon)^n}$$

Since the integrand is only a function of the modulus of the Euclidean 4-vector, we can perform the angular integral and transform the 4-dim. into a 1-dim integral:

$$\int d^4 q_E \, f(q_E^2) = \pi^2 \int_0^\infty dz \, f(z)$$

$$\Rightarrow I_1(a) = \frac{-i}{16\pi^2} \int_0^\infty \frac{z\,dz}{z-a} \quad \text{which is infinite.}$$

The simplest way to regularize the integral is to introduce an upper limit of integration. We then have,

$$I_1(a) = \frac{-i}{16\pi^2} \int_0^\Lambda \frac{z\,dz}{z-a} = \frac{-i}{16\pi^2} \left[\Lambda + a \ln\left(\frac{\Lambda - a}{-a}\right) \right]$$

All other integrals ($n \geq 2$) can be obtained from this by differentiation with respect to a. $I_2(a)$ is logarithmically divergent, but then all the others are finite.

2. Dimensional regularization.

This was introduced by Bollini and Giambiagi in 1972 and soon after, and independently as well as more extensively by 't Hooft and Veltman. The idea is the following:

One rewrites the Euclidean integral $I_n(a)$ as an integral in d instead of 4 dimensions:

$$I_n(a) = \int \frac{d^d q_E}{(2\pi)^d} \frac{1}{(-q_E^2 + a)^n}$$

Of course the integral only makes sense for $d \in \mathbb{N}$, but once we have done the calculation and have the result expressed in terms of d we can analytically continue the function to any value of d. In particular we first need to calculate the angular integral for a generic value of d:

$$\int d\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$$

The radial integral goes as follows (to simplify the treatment $-q_E^2 \to q^2$)

$$\int_0^\infty dq \, \frac{q^{d-1}}{(q^2+a)^n} = \frac{1}{2}\int_0^\infty dq^2 \, \frac{(q^2)^{d/2-1}}{(q^2+a)^n}$$

$$x = \frac{a}{q^2+a} \quad ; \quad q^2 = \frac{a}{x} - a = a\left(\frac{1-x}{x}\right)$$

$$q^2 = 0 \Rightarrow x = 1 \; ; \; q^2 = \infty \Rightarrow x = 0$$

$$\frac{dq^2}{dx} = a\left(-\frac{1}{x} - \frac{1-x}{x^2}\right) = -\frac{a}{x}\left(\frac{x+1-x}{x}\right) = -\frac{a}{x^2}$$

$$dq^2 = -a\frac{dx}{x^2}$$

$$(q^2)^{d/2-1} = a^{d/2-1}(1-x)^{d/2-1} \cdot x^{1-d/2}$$

$$(q^2+a)^{-n} = a^{-n} \cdot x^n$$

$$\Rightarrow \int_0^\infty dq \, \frac{q^{d-1}}{(q^2+a)^n} = \frac{1}{2} a^{d/2-n} \cdot \int_0^1 dx \, x^{n-d/2-1} (1-x)^{d/2-1}$$

$$= \frac{1}{2} a^{d/2-n} \frac{\Gamma(n-d/2)\Gamma(d/2)}{\Gamma(n)}$$

$$\Rightarrow \int \frac{d^dq}{(2\pi)^d} \frac{1}{(q^2+a)^n} = \frac{\pi^{d/2}}{(2\pi)^d} \cdot a^{d/2-n} \frac{\Gamma(n-d/2)}{\Gamma(n)} = (4\pi)^{-d/2}\left(\frac{1}{a}\right)^{n-d/2} \frac{\Gamma(n-d/2)}{\Gamma(n)}$$