Result of the calculation to $O(g^2)$.

$$-i\tilde{\Pi}^f(p^2) = g^2 \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx \, \frac{1}{[k^2+p^2x(1-x)-m^2+i\varepsilon]^2}$$

$$= \frac{-ig^2}{16\pi^2} \int_0^1 dx \, \ln\left(m^2-p^2x(1-x)-i\varepsilon\right) + \dots$$

where the ellipsis stands for terms which cancel out in a convergent combination.

The quantity of interest, which is finite after having imposed the renormalization conditions is $\tilde{\Pi}_r(p^2) = \tilde{\Pi}^f(p^2) - \tilde{\Pi}^f(\mu^2) - (p^2-\mu^2)\left.\frac{d\tilde{\Pi}^f(p^2)}{dp^2}\right|_{\mu^2}$

$$\Rightarrow \tilde{\Pi}_r(p^2) = \frac{g^2}{16\pi^2}\int_0^1 dx \left[\ln\left(\frac{m^2-p^2x(1-x)-i\varepsilon}{m^2-\mu^2x(1-x)}\right) + \frac{(p^2-\mu^2)\,x(1-x)}{m^2-\mu^2x(1-x)}\right]$$

where I have removed the $i\varepsilon$ from the denominator in the log and in the second term because, assuming $\mu^2<4m^2$, $m^2-\mu^2x(1-x)$ will never vanish.

Let us try to understand the role of $i\varepsilon$ and how it determines the sign of the imaginary part of the log. Since the denominator is always positive, we can ignore it and look only at the numerator:

$$m^2-p^2x(1-x)$$

when this becomes negative the log develops an imaginary part. Since the maximum of $x(1-x)$ is $\frac{1}{4}$, the argument of the log is negative only if $p^2 \geq 4m^2$. In that case we can use:

$\log(-x \pm i\varepsilon) = \pm i\pi$, from which we get

$y \equiv x - 1/2$, $\int_0^1 dx \to \int_{-1/2}^{1/2} dy$; $x(1-x) = (y+1/2)(\frac{1}{2}-y) = \frac{1}{4} - y^2$

$\Rightarrow \quad x(1-x) = \frac{m^2}{p^2} \Rightarrow \quad y^2 = \frac{1}{4} - \frac{m^2}{p^2} \Rightarrow \quad y = \pm\frac{1}{2}\sqrt{1-\frac{4m^2}{p^2}} \equiv \pm\frac{1}{2}\beta(p^2)$

$$\mathrm{Im}\,\tilde{\Pi}_r(p^2) = \frac{-g^2}{16\pi^2}\int_{-1/2\beta}^{1/2\beta} dy\, \pi = \frac{-g^2}{16\pi}\beta(p^2)$$

Remember:

$\mathrm{Im}\left[-i\tilde{D}_r(p^2)\right] = -\pi\sigma(p^2)$ for $p^2 > \mu^2$ (in this case $4m^2$)

[$-i\tilde{D}_r(p^2) = -i\tilde{D} + i(-i\tilde{D})(-i\tilde{\Pi}_r(p^2))(-i\tilde{D})$

$\mathrm{Im}\left[-i\tilde{D}_r(p^2)\right] = 0 + (-i\tilde{D})^2 \cdot \mathrm{Im}\,\tilde{\Pi}_r(p^2)$

and since $\sigma(p^2)$ was positive, we conclude that the imaginary part of $\Pi_r(p^2)$ has to be negative, which is what we found.]

$$-i\tilde{D}_r(p^2) = \frac{1}{p^2-\mu^2-\tilde{\Pi}_r(p^2)} \qquad \mathrm{Im}\left(\frac{1}{z}\right) = \frac{-y}{x^2+y^2} \quad \text{if } z = x+iy$$

In our case $x = p^2-\mu^2-\mathrm{Re}\,\tilde{\Pi}_r(p^2)$, $y = -\mathrm{Im}\,\Pi_r(p^2)$

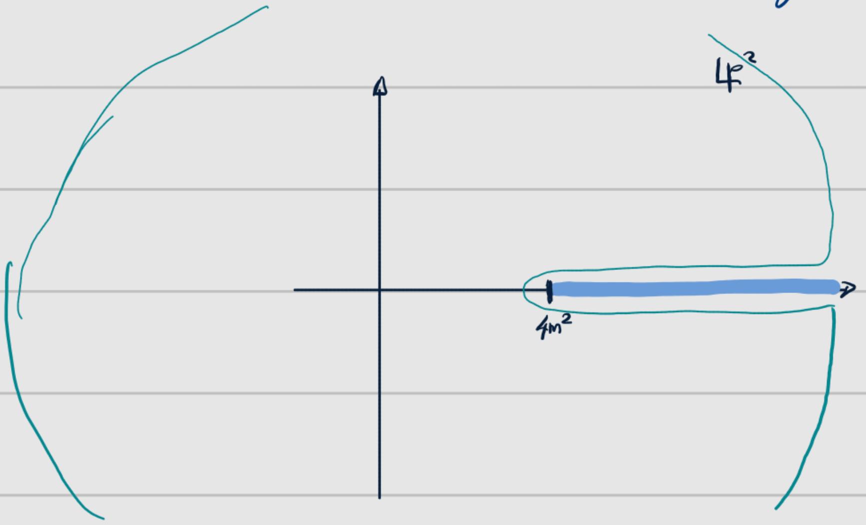
$$\mathrm{Im}\left[-i\tilde{D}_r(p^2)\right] = \frac{\mathrm{Im}\,\Pi_r(p^2)}{\left(p^2-\mu^2-\mathrm{Re}\,\tilde{\Pi}_r(p^2)\right)^2 + \left(\mathrm{Im}\,\Pi_r(p^2)\right)^2}$$

which leads to the same conclusion.

Note that if I write

$$-i\tilde{D}_r(p^2) = \frac{p^2-\mu^2-\tilde{\Pi}_r^*(p^2)}{|p^2-\mu^2-\tilde{\Pi}_r(p^2)|^2}$$

it is clear that the discontinuity is completely due to $-\tilde{\Pi}_r(p^2)$, and that this is otherwise an analytic function.



If $f(z)$ is analytic, then I can write

$$f(s) = \frac{1}{2\pi i}\oint dz \, \frac{f(z)}{z-s}$$

where the integration path has to be in the analyticity domain of f and s has to lie inside the closed contour. The formula can be applied to $\tilde{\Pi}_r(p^2)$ for the integration path shown on the plot above. Sending the circle to infinity and the path around the cut infinitesimally close to the real axis we end up with (since $\mathrm{Im}\,\tilde{\Pi}_r(p^2) = \mathrm{Im}\,\tilde{\Pi}^f(p^2)$ I'll write the formula for $\tilde{\Pi}^f(p^2)$):

$$\tilde{\Pi}^f(p^2) = \frac{1}{2\pi i}\int_{4m^2}^{\infty} ds \, \frac{\mathrm{Disc}\,\tilde{\Pi}^f(s)}{s-p^2} = \frac{1}{\pi}\int_{4m^2}^{\infty} ds \, \frac{\mathrm{Im}\,\tilde{\Pi}^f(s)}{s-p^2}$$

If we subtract the value at $p^2=\mu^2$ we obtain:

$$\tilde{\Pi}^f(p^2) - \tilde{\Pi}^f(\mu^2) = \frac{1}{\pi}\int_{4m^2}^{\infty} ds\, Im\,\tilde{\Pi}^f(s)\left(\frac{1}{s-p^2} - \frac{1}{s-\mu^2}\right)$$

$$= \frac{(p^2-\mu^2)}{\pi}\int_{4m^2}^{\infty} ds\, \frac{Im\,\tilde{\Pi}^f(s)}{(s-\mu^2)(s-p^2)}$$

similarly one can obtain $\quad$ (remember, $Im\,\tilde{\Pi}^f(p^2) = -\frac{g^2}{16\pi}\beta(p^2)$)

$$\tilde{\Pi}_r(p^2) = -\frac{(p^2-\mu^2)^2 g^2}{16\pi^2}\int_{4m^2}^{\infty} ds\, \frac{\beta(s)}{(s-\mu^2)^2(s-p^2)}$$

Is this equal to the expression above written as an integral over x of a log and a rational function? Since the two expressions fulfill the same analytic properties, they must coincide.

Let us now check whether we understand not only the sign, but also the exact expression of the imaginary part. From the definition of $\sigma(q^2)$

$$\sigma(q^2)\,\theta(q^0) = \sum_n{}' (2\pi)^3 \delta^4(q-p_n)\, |\langle n|\phi(0)|0\rangle|^2$$

we can conclude that, at $O(g^2)$, the only contribution comes from a 2-nucleon state, $n = 2N$, in which case $|\langle 2N|\phi(0)|0\rangle|^2 = \frac{g^2}{(p^2-\mu^2)^2}$

To get the self-energy we can drop the factor $\frac{1}{(p^2-\mu^2)^2}$ and keep only g^2.

The only thing we need to calculate is the sum over all possible states which is, in the present case, an integral over the phase space:

$$\int \frac{d^3p_1}{(2\pi)^3 2\omega_1}\frac{d^3p_2}{(2\pi)^3 2\omega_2}(2\pi)^4\delta^4(q-(p_1+p_2)) = \int\frac{d^3p_1}{(2\pi)^2 2\omega_1}\int d^4p_2\, \delta(p_2^2-m^2)\,\delta^4(q-(p_1+p_2))$$

$$= \int \frac{d^3p_1}{(2\pi)^2 2E_1}\, \delta\left((q-p_1)^2 - m^2\right) = \int \frac{dp_1 \cdot p_1^2\, d\Omega_1}{(2\pi)^2\, 2E_1}\, \delta\left(q^2 - 2p_1\cdot q + \cancel{m^2} - \cancel{m^2}\right)$$

In the CoM frame $p_1 \cdot q = E_1 q^0$, $q^0 = \sqrt{q^2} \Rightarrow \delta(q^2 - 2E_1\sqrt{q^2}) = \delta(q^2 - 2E_1\sqrt{q^2}) =$

$$= \delta\left(2\sqrt{q^2}\left(\frac{\sqrt{q^2}}{2} - E_1\right)\right) = \frac{1}{2\sqrt{q^2}}\, \delta\left(E_1 - \frac{\sqrt{q^2}}{2}\right)$$

$$\int d\Omega_1 = 4\pi \;; \quad \int \frac{dp_1\, p_1^2}{E_1} = \int \frac{dE_1 \cdot \cancel{E_1}\, |\vec{p}_1|}{\cancel{E_1}} \qquad |\vec{p}_1| = \sqrt{\frac{q^2}{4} - m^2}$$

$$\Rightarrow \int \frac{d^3p_1}{(2\pi)^2 2E_1}\, \delta\left((q-p_1)^2 - m^2\right) = \frac{d\Omega}{8\pi^2}\, \frac{1}{2\sqrt{q^2}} \sqrt{\frac{q^2}{4} - m^2} = \frac{1}{16\pi^2}\, \frac{1}{2}\sqrt{1 - \frac{4m^2}{q^2}}\, d\Omega = \frac{1}{8\pi}\beta(q^2)$$

To obtain $\sigma(q^2)$ we need to divide by 2π, so we get $\frac{1}{16\pi^2}\beta(q^2)$, which is exactly what we obtained from the explicit calculation.

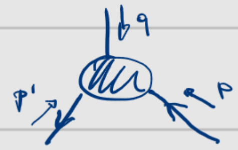
Coupling constant renormalization.

We now consider the last counterterm, F:

$$\mathcal{L} = \ldots + F\, \varphi^* \varphi\, \phi$$

which is clearly related to a 3-point Green's function.

Let us define a 1PI graph:

$$-i\tilde{\Gamma}_r(p^2, p'^2, q^2) =$$ 

Up to $\mathcal{O}(g^3)$ we have the following contributions

$$-i\tilde{\Gamma}_r(p^2, p'^2, q^2) =$$  +  + 

$$\tilde{\Gamma}_r = g + \tilde{\Gamma}^f(p^2, p'^2, q^2) + F$$

$$\tilde{\Gamma}^f(p^2, p'^2, q^2) = g(-ig)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-\mu^2+i\varepsilon} \cdot \frac{i}{(k+p)^2-M^2+i\varepsilon} \cdot \frac{i}{(k-p')^2-M^2+i\varepsilon}$$

$$= ig^3 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2-\mu^2+i\varepsilon} \cdot \frac{1}{(k+p)^2-M^2+i\varepsilon} \cdot \frac{1}{(k-p')^2-M^2+i\varepsilon}$$

This integral is finite. We now need to impose a condition that fixes the coupling constant to its physical value. The exact way how one does this is arbitrary. We follow Coleman and impose:

$$\tilde{\Gamma}_r(m^2, m^2, \mu^2) = g$$

$$\Rightarrow \tilde{\Gamma}^f(m^2, m^2, \mu^2) + F = 0$$

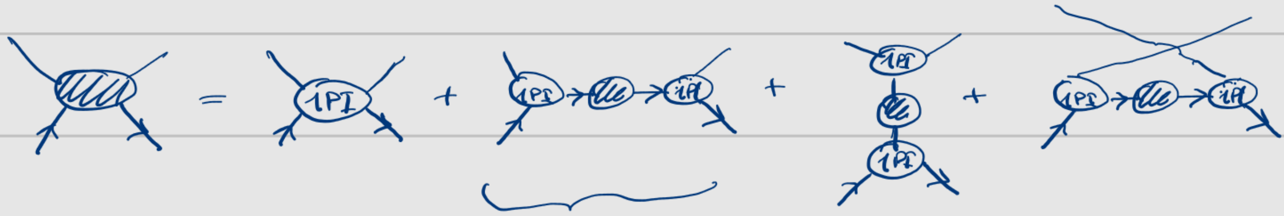
Note that the if we want to satisfy $p+p'+q=0$

$$(p+p'+q)^2 = 2m^2+\mu^2 + 2p\cdot(p'+q) + 2p'q = 0$$

$$q^2 = (p+p')^2 = 4E^2 \text{ in the CoM}$$

$$\text{where } E^2 = \vec{p}^{\,2} + m^2$$

if $\mu^2 < 4m^2$ (as we have assumed) all conditions (3 mass-shell and 1 momentum conservation) cannot be satisfied. However, one can reach that point by analytic continuation. Why this condition has been chosen can be seen with the help of a scattering process. For example $\phi N \to \phi N$



$$\frac{-ig^2}{s-m^2}$$

all other contributions are regular at $s=m^2$. So the coupling constant is defined as the residue of the pole at $s=m^2$ in the ϕN scattering process. The point $s=m^2$ is outside the physical region but it can be reached by analytic continuation, for example starting from data.

Renormalizability.

In the theory we are considering we have encountered only two divergent diagrams:

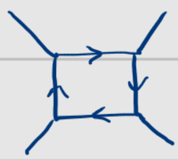
the tadpole

the bubble

The third one we have considered

the triangle is finite.

Clearly, also more complicated one-loop diagrams like



will also be finite.

How can we easily decide whether a diagram is finite or not?

For each loop we get an integral d^4k

each propagator $\frac{1}{p^2-\mu^2/m^2}$

each vertex gives a constant, which is irrelevant. The superficial degree of divergence is given by

$$D = 4L - 2I$$

where L is the number of loops and I of internal lines. Vertices play a role in the counting of the total number of lines, since the number of external lines E plus twice the number of internal lines must be equal to $3 \cdot V$, with V the number of vertices:

$$3V = E + 2I$$

$$\Rightarrow \quad D = 4L - 3V + E$$

$E = 1,\ L = 1,\ V = 1 \Rightarrow D = 4 - 3 + 1 = 2$

$E = 2,\ L = 1,\ V = 2 \Rightarrow D = 4 - 6 + 2 = 0$

But we can go further: The number of loops in a diagram is given by

$$L = I - V + 1$$

because each propagator comes with an integral but each vertex with a δ-function for momentum conservation, from which we have to subtract one for the overall momentum conservation for the whole diagram.

Putting everything together we get

$$D = 4I - 4V + 4 - 2I = 2I - 4V + 4 = 3V - E - 4V + 4$$

$$= 4 - V - E$$

Clearly, as we increase either the number of external lines or of the vertices (and therefore of the loops, for E fixed), the superficial degree of divergence decreases. Let us look again at known diagrams:

$D = 4 - 1 - 1 = 2$

$D = 4 - 2 - 2 = 0$

$D = 4 - 3 - 3 = -2$

From this we conclude that the number of divergent diagrams in this theory is finite: only the two we have already encountered are divergent. All other ones are finite.

Theories with this property are called <u>super-renormalizable</u>.

To simplify our further discussion let us consider a scalar theory with only one type of field and just a power of the field as interaction:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{n!}\phi^n$$

In this case we need to modify the equation

$$3V = E + 2I \implies nV = E + 2I$$

which leads to: $\quad D = 4 + (n-4)V - E$

We have considered already the case $n=3$, let us go to $n=4$.

In this case

$$D = 4 - E$$

 $\quad D = 4-2 = 2$

 $\quad D = 4-4 = 0$

 $\quad D = 4-6 = -2$

In this theory only a finite number of amplitudes are divergent, but if we increase the number of loops, they will still be divergent.

  are both divergent.

  " "

$\Rightarrow$ I need to introduce counterterms for the mass and wave-funct. normalization, as well as for the coupling constant λ, and they will be divergent at each loop order, but they are all what is needed to make the theory finite.

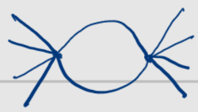
Such a theory is called <u>renormalizable</u>.

If we move to $n=5$ or 6, things get even worse:

$$D = 4 + 2V - E$$

Amplitudes with any number of external legs can be made divergent:

$D = 4+4-4 = 4$



$D = 4+4-8 = 0$ logarithmically div.

If I start with ϕ^6 I discover that I need a counterterm of the type ϕ^4 to make the first diagram above finite. The same is true for the second diagram which requires a counterterm of the type ϕ^8. And so on...

These theories are called <u>non-renormalizable</u>.

<u>Role of spacetime dimension.</u>

If we change the spacetime dimension from 4 to d we need to modify the relations above.

In particular: $D = dL - 2I$

$L = I - V + 1$ and $nV = E + 2I$ remain unchanged

$$D = dI - dV + d - 2I = (d-2)I - dV + d$$

$$= (d-2)\left(\frac{n}{2}V - \frac{E}{2}\right) - dV + d$$

$$\Rightarrow \quad D = d + \left(n\left(\tfrac{d-2}{2}\right) - d\right)V - \left(\tfrac{d-2}{2}\right)E$$

In $d=6$, $n=4$ we get $D = 6 + (8-6)V - 2E$
$= 6 + 2V - 2E = 2(3+V-E)$

which is non-renormalizable.

But in $d=6, n=3$ gives $D = 6 + (6-6)V - 2E = 6-2E$

which is renormalizable.

The same conclusion can be reached by a dimensional analysis:

$$S = \int d^dx \; \mathcal{L} \qquad \mathcal{L} = \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2$$

$$[\phi] = \frac{d-2}{2} \Rightarrow [\lambda\phi^n] = d \Rightarrow [\lambda] = d - n\frac{d-2}{2}$$

which is exactly minus the coeff. multiplying V in the formula above.

The point is that an amplitude with E external legs has dimension

$d - E\left(\frac{d-2}{2}\right)$ (as coming from a term $g\,\phi^E$ in the Lagrangian $\Rightarrow [g] = d - E\left(\frac{d-2}{2}\right)$)

and that a diagram with V λ-vertices diverges as $\lambda^V \cdot \Lambda^D$

$$\Rightarrow D + V\left(d - n\tfrac{d-2}{2}\right) = d - E\left(\tfrac{d-2}{2}\right)$$ which coincides with what we had above.

$\Rightarrow$ If the coupling constant has negative mass dimension, the theory is going to be non-renormalizable.