In our previous discussions we excluded the case $\mu^2 > 4m^2$, but we now want to discuss it explicitly. In view of our calculation of the imaginary part of the self-energy it is clear that this will be nonzero also at $p^2 = \mu^2$:

$$\mathrm{Im}\,\tilde{\Pi}_r(p^2) = -\frac{1}{2}\left|\tilde{D}_r(p^2)\right|^{-2} \sum_n{}' \left|\langle n|\phi(0)|0\rangle\right|^2 (2\pi)^4 \delta^4(p-q_n)$$

With a nonzero imaginary part, imposing as renormalization condition

$$\tilde{\Pi}_r(\mu^2) = 0 \quad \text{and} \quad \frac{d\tilde{\Pi}_r}{dp^2}(\mu^2) = 0$$

would have as consequence complex counterterms and a non-hermitian Hamiltonian. But since a decaying particle is not an asymptotic state for which the concept of mass has a clear unambiguous meaning, the renormalization conditions can be chosen more arbitrarily for an unstable particle. Coleman adopts the following convention:

$$\mathrm{Re}\,\tilde{\Pi}_r(\mu^2) = 0 \quad \text{and} \quad \mathrm{Re}\,\frac{d\tilde{\Pi}_r}{dp^2}(\mu^2) = 0$$

arguing that they are continuous even as one moves from the stable to the unstable case.

Consider a power series in g^2 for the inverse propagator:

$$\left[-i\tilde{D}_r(p^2)\right]^{-1} = p^2 - \mu^2 - \tilde{\Pi}_r(\mu^2) - \underbrace{(p^2-\mu^2)\frac{d\tilde{\Pi}_r}{dp^2}(\mu^2)}_{=\mathcal{O}(g^4)} + \mathcal{O}(g^4)$$

$$\Rightarrow \left[-i\tilde{D}_r(p^2)\right]^{-1} = p^2 - \mu^2 - i\,\mathrm{Im}\,\tilde{\Pi}_r(\mu^2)$$

But at order g^2 we can calculate $\mathrm{Im}\,\tilde{\Pi}_r(\mu^2)$ and we get:

$$\mathrm{Im}\,\tilde{\Pi}_r(p^2) = -\frac{1}{2}\left|\tilde{D}_r(p^2)\right|^{-2} \sum_n \left|-ig\frac{i}{p^2-\mu^2+i\varepsilon}\right|^2 (2\pi)^4 \delta^4(k+k'-p)$$

$$= -\frac{1}{2} g^2 \int \frac{d^3k}{(2\pi)^3 2E_k} \frac{d^3k'}{(2\pi)^3 2E_{k'}} \; (2\pi)^4 \delta^4(k+k'-p)$$

This is nothing but the squared matrix element for the decay $\phi \to N\bar{N}$ (modulo a factor $-\frac{1}{2}$) which we would have calculated from the Feynman diagram

$-ig$

The decay rate is given by:

$$\Gamma = \frac{1}{2M} \sum_{\substack{\text{final}\\ \text{states}}} |A_{fi}|^2 D$$

where D is the invariant phase space differential:

$$D = (2\pi)^4 \delta^4(p_i - p_f) \prod_f \frac{d^3p_f}{(2\pi)^3 2E_f}$$

So, we conclude that:

$$\lim_{p^2 \to \mu^2} |\tilde{D}_r(p^2)|^{-2} \sum_n |\langle n|\phi(0)|0\rangle|^2 (2\pi)^4 \delta^4(p-q_n) = \sum_f \int |A_{fi}|^2 D = 2M\Gamma$$

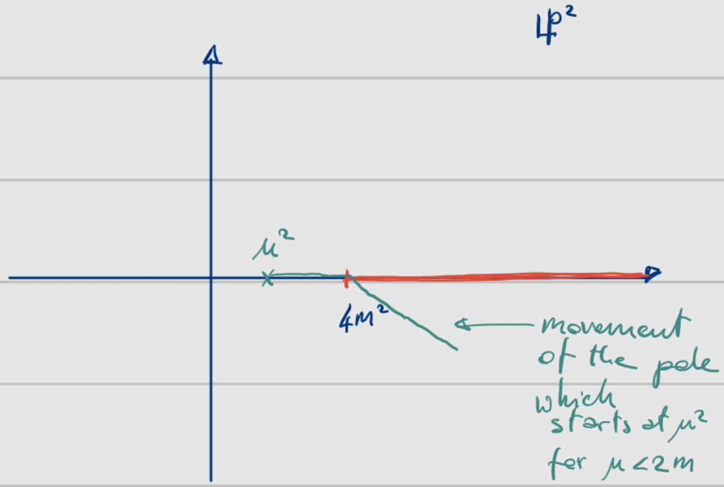
$$\Rightarrow \operatorname{Im} \tilde{\Pi}_r(\mu^2) = -\frac{1}{2}(2\mu\Gamma) = -\mu\Gamma$$

and from this

$$\left[-i\tilde{D}_r(p^2)\right]^{-1} = p^2 - \mu^2 - i \operatorname{Im} \tilde{\Pi}_r(\mu^2) = p^2 - \mu^2 + i\mu\Gamma + O(g^4)$$

$$= p^2 - \left(\mu - \frac{i}{2}\Gamma\right)^2 + O(g^4)$$

This shows that the pole that was on the real axis gets now shifted below the real axis. Since this is done continuously starting from above the real axis, it means that we end up on the second Riemann sheet:



Breit-Wigner formula.

Consider a propagator with a pole shifted by an imaginary part:

$$\tilde{D}_\Gamma(p^2) = \frac{i}{p^2 - \mu^2 + i\mu\Gamma} \qquad \text{with } \mu, \Gamma \in \mathbb{R}$$

supposing that this is just the form of the two-point function in this theory, without knowing what is the origin of Γ (a particle which decays or just a short-lived resonance state).

Imagine now to have the Lagrangian $\mathcal{L}$ which is responsible for the form of $\tilde{D}_\Gamma(p^2)$ above. Add to this a perturbation of the form:

$$\mathcal{L} \to \mathcal{L} + \rho(x)\phi(x)$$

$$\text{with} \quad \rho(x) = \lambda \delta^4(x) .$$

To lowest order in λ, the amplitude for creating a state $|n\rangle$ out of the vacuum is given by:

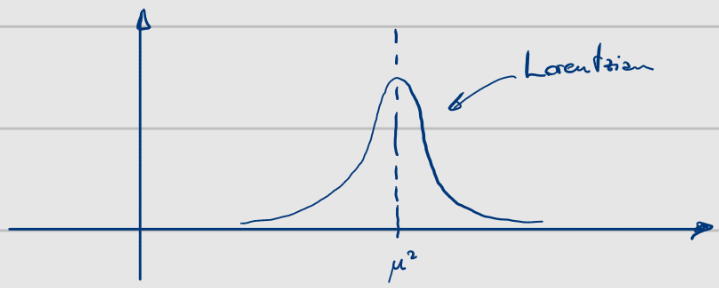
$$A_{vac \to n} \propto \lambda \langle n|\phi(0)|0\rangle + \mathcal{O}(\lambda^2)$$

and the corresponding probability for creating such a state:

$$P(k) \propto \lambda^2 \sum_n |\langle n|\phi(0)|0\rangle|^2 (2\pi)^4 \delta^4(k-p_n) + \mathcal{O}(\lambda^3)$$

Near $k^2=\mu^2$ the formula exactly corresponds to the imaginary part of the propagator:

$$P(k) \propto \lambda^2 \sigma(k^2) \propto \underbrace{-\lambda^2 \operatorname{Im}\left(-i\tilde{D}_F(k^2)\right)}_{= \lambda^2 \frac{\mu\Gamma}{(k^2-\mu^2)^2+\mu^2\Gamma^2}} + \mathcal{O}(\lambda^3)$$



$k^2-\mu^2=\pm\mu\Gamma \Rightarrow$ full width at half maximum is $2\mu\Gamma$, but if we convert the expression to depending on energy in the rest frame of the decaying particle:

$$k=(E,\vec{0}) \quad \Rightarrow \quad k^2-\mu^2 = E^2-\mu^2 = (E+\mu)(E-\mu) \sim 2\mu(E-\mu)$$

we end up with

$$P(E) \propto \frac{\lambda^2\mu\Gamma}{4\mu^2(E-\mu)^2+\mu^2\Gamma^2} = \frac{\lambda^2\Gamma/4\mu}{(E-\mu)^2+\frac{\Gamma^2}{4}}$$

which is the usual Breit-Wigner denominator.

Resonances.

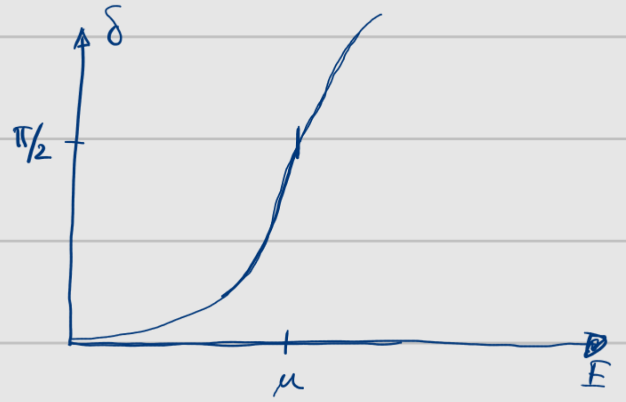
Decay amplitude $\propto$ T-matrix element $\quad S = 1 + 2iT$

$$1 - \frac{2i\mu\Gamma}{p^2-\mu^2+i\mu\Gamma} = \frac{p^2-\mu^2-i\mu\Gamma}{p^2-\mu^2+i\mu\Gamma} = \text{complex unit vector}$$

$S = e^{2i\delta}$, indeed unitary. Unitary S-matrix in the presence of a single channel. δ is called the phase shift.

$$p^2-\mu^2 \mp i\mu\Gamma \equiv \rho e^{\pm i\delta} \qquad S = e^{2i\delta}$$

$$\Rightarrow \quad \tan\delta = \frac{-\mu\Gamma}{p^2-\mu^2} \simeq \frac{-\mu\Gamma}{2\mu(E-\mu)} = \frac{-\Gamma/2}{E-\mu}$$



typical behaviour of the phase shift in the presence of a resonance.

Exponential decay law.

Time evolution of a state in QM $\quad e^{-iEt}$

Breit-Wigner formula: pole shifted from $E \to E_R - i\Gamma/2$

$$\hookrightarrow e^{-iE_Rt}e^{-\Gamma t/2}$$

A more detailed derivation is the subject of the new exercise series.

We will follow the derivation by Coleman.

He introduces a source coupled to the field ϕ, in order to represent the process of creation or absorption (i.e. measurement) of the meson ϕ.

$$\mathcal{L} \to \mathcal{L} + f(x)\phi(x) \qquad f(x) \in \mathbb{R}$$

which implies $\tilde{f}^*(k) = \tilde{f}(-k)$

Moreover we want $f(x)$ to be localised around the origin.

Creating the meson at the origin : $\int d^4x \, f(x)\phi(x)|0\rangle$

Absorbing the meson at a distance y : $x' = x + y$

Since the source is different from zero only around the origin, the amplitude I am interested in should be written as

$$A(y) = \langle 0| \int d^4x' d^4x \, f(x'-y)\phi(x') f(x)\phi(x)|0\rangle$$

$$= \int d^4x' d^4x \, f(x'-y) f(x) \langle 0| T\phi(x')\phi(x)|0\rangle$$

since $y^0 > x^0$.