Recapitulation about spinors and their transformation properties

Dirac fermions transform according to the spinor representation of the Lorentz group: $D^{(1/2,0)}(\Lambda) \oplus D^{(0,1/2)}(\Lambda)$

$$\psi = \begin{pmatrix} u_+ \\ u_- \end{pmatrix}$$ where u_+ transform according to $D^{(1/2,0)}$
u_- " " $D^{(0,1/2)}$

Generators of rotations in this representations:

$$\vec{L} = \frac{1}{2}\vec{\sigma} = \left(\vec{J}^{(+)} + \vec{J}^{(-)}\right)$$

Generators of boosts:

$$\vec{M} = \pm \frac{i}{2}\sigma = -i\left(\vec{J}^{(+)} - \vec{J}^{(-)}\right)$$

For finite transformations we have:

Rotation by the angle θ around the $\hat{n}$ axis.

$$R(\hat{n}\theta): \quad u_\pm \longrightarrow e^{-\frac{i}{2}\vec{\sigma}\cdot\hat{n}\theta} u_\pm$$

Boost along the $\hat{\alpha}$ axis with speed $v = \tanh\phi$

$$A(\hat{\alpha}\phi): \; u_\pm \longrightarrow e^{\pm\frac{i}{2}\vec{\sigma}\cdot\hat{\alpha}\phi} u_\pm$$

$u_\pm$ are Weyl spinors. Parity switches them:

$$P u_\pm = u_\mp$$

Dirac Lagrangian: $\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi$

Plane wave, positive-energy solutions: $\psi(\vec{x},t) = u_{\vec{p}}\, e^{-ipx}$

where $u_{\vec{p}}$ has to satisfy: $\left(E_{\vec{p}} - \vec{\alpha}\cdot\vec{p}\right)u_{\vec{p}} = \beta m u_{\vec{p}}$

with $\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}$ and $\beta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$

For $\vec{p}=0$ the Dirac equation becomes

$$u_{\vec{0}} = \beta\, u_{\vec{0}}$$

which admits two linearly independent solutions

$$u^{(1)}_{\vec{0}} = \sqrt{2m}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad \text{and} \quad u^{(2)}_{\vec{0}} = \sqrt{2m}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

After a boost they become:

$$u^{(1)}_{\vec{p}} = \begin{pmatrix}\sqrt{E_{\vec{p}}+m}\\0\\\sqrt{E_{\vec{p}}-m}\\0\end{pmatrix} \quad \text{and} \quad u^{(2)}_{\vec{p}} = \begin{pmatrix}0\\\sqrt{E_{\vec{p}}+m}\\0\\-\sqrt{E_{\vec{p}}-m}\end{pmatrix}$$

Plane-wave, negative-energy solutions: $\psi(\vec{x},t) = v_{\vec{p}}\, e^{ipx}$

Dirac eq.: $\left(E_{\vec{p}} - \vec{\alpha}\cdot\vec{p}\right) v_{\vec{p}} = -\beta m\, v_{\vec{p}}$

$\vec{p}=0$: $\quad v_{\vec{0}} = -\beta\, v_{\vec{0}}$

$$v^{(1)}_{\vec{0}} = \sqrt{2m}\begin{pmatrix}0\\0\\0\\1\end{pmatrix} \; ; \quad v^{(2)}_{0} = \sqrt{2m}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$$

and after a boost:

$$v^{(1)}_{\vec{p}} = \sqrt{2m}\begin{pmatrix}0\\-\sqrt{E_{\vec{p}}-m}\\0\\\sqrt{E_{\vec{p}}+m}\end{pmatrix} \; ; \quad v^{(2)}_{\vec{p}} = \begin{pmatrix}\sqrt{E_{\vec{p}}-m}\\0\\\sqrt{E_{\vec{p}}+m}\\0\end{pmatrix}$$

Parity transformation.

$P:\; \psi(\vec{x},t) \longrightarrow \beta\,\psi(-\vec{x},t)$ (in general there could be a phase factor, but we omit it here).

For ψ as operators we have:

$$U_P^{+}\,\psi(\vec{x},t)\,U_P = \beta\,\psi(\vec{x},t) \Rightarrow \quad U_P^{+}\begin{Bmatrix} b^{(r)}_{\vec{p}} \\ c^{(r)+}_{\vec{p}}\end{Bmatrix} U_P = \begin{Bmatrix} b^{(r)}_{-\vec{p}} \\ -c^{(r)+}_{-\vec{p}}\end{Bmatrix}$$

$$U_P^{+}\begin{Bmatrix} b^{(r)+}_{\vec{p}} \\ c^{(r)}_{\vec{p}}\end{Bmatrix} U_P = \begin{Bmatrix} b^{(r)+}_{-\vec{p}} \\ -c^{(r)}_{-\vec{p}}\end{Bmatrix}$$

END of Recapitulation

Renormalization of Dirac fermions.

To discuss renormalization for spin-1/2 particles we take model 3 but this time with spin-1/2 nucleons.

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m_0)\psi + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\mu_0\phi^2 + g_0\bar{\psi}i\gamma_5\psi\phi$$

bare masses and couplings.

We take ϕ to be a pseudoscalar, so that the theory is parity-invariant.

Because of this we don't need to worry about a vev for ϕ:

$$\langle 0|\phi(x)|0\rangle = \langle 0|e^{ipx}\phi(0)e^{-ipx}|0\rangle = \langle 0|\phi(0)|0\rangle = 0$$

Matrix element between vacuum and 1P-state:

$$\langle 0|\phi(0)|p\rangle = \sqrt{Z_3} \qquad \text{with } Z_3 \in \mathbb{R} \text{ and } Z_3 > 0$$

$\Rightarrow$ redefine $\phi'(x) = \frac{1}{\sqrt{Z_3}}\phi(x)$.

We did this already for our only-scalar model 3.

Now we want to do the same for ψ and for this we need to look at the matrix element:

$$\langle 0|\psi(x)|r,p\rangle \qquad \text{where } r \text{ is the spin state.}$$

($\bar{\psi}$ is not independent as it is related to ψ by charge conjugation).

The state $|r,p\rangle$ can be obtained from a zero-momentum state by a boost:

$$|r,p\rangle = U(\Lambda)\,|r,(m,\vec{0})\rangle$$

If we take $r = +1/2$, we can obtain $r = -1/2$ by acting with J_-

$$|-1/2;(m,\vec{0})\rangle = (J_x - iJ_y)\,|1/2;(m,\vec{0})\rangle$$

$\Rightarrow$ we only need to study $|1/2;(m,\vec{0})\rangle$, all the rest can be obtained by Lorentz transformation: $\langle 0|\psi(0)|1/2;(m,\vec{0})\rangle$

For a free field theory we have (we set $r=1$ for spin $+1/2$ and $r=2$ for $-1/2$):

$$\langle 0|\psi_{\text{free}}(0)\,|1/2;(m,\vec{0})\rangle = u^{(1)}_{\vec{0}} \qquad = \sqrt{2m}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

For the interacting theory we can use the transformation properties under rotations to determine the matrix element:

$$\underbrace{\langle 0|e^{i\theta J_z}}_{\langle 0|}\psi(0)\underbrace{e^{-i\theta J_z}|1/2;(m,\vec{0})\rangle}_{e^{-i/2\,\theta}|1/2;(m,\vec{0})\rangle}$$

but we also know how the rotation acts on the field:

$$e^{-iL_z\theta}\langle 0|\psi(0)|1/2;(m,\vec{0})\rangle$$

with $L_z = \frac{1}{2}\begin{pmatrix}\sigma_3 & 0\\ 0 & \sigma_3\end{pmatrix}$

For the two results to be equal we must have

$$\langle 0|\psi(0)|1/2;(m,\vec{0})\rangle = \sqrt{2m}\begin{pmatrix}a\\0\\b\\0\end{pmatrix}$$

but then we can use parity acting either on the states:

$$\langle 0|U_P^{\dagger}\psi(0)U_P|1/2;(m,\vec{0})\rangle = \langle 0|\psi(0)|1/2;(m,\vec{0})\rangle$$

or on the field:

$$\langle 0|U_P^{\dagger}\psi(0)U_P|1/2;(m,\vec{0})\rangle = \underset{\substack{\parallel\\ \gamma_0}}{\beta}\langle 0|\psi(0)|1/2;(m,\vec{0})\rangle$$

in the standard representation

We conclude that: $\begin{pmatrix}a\\0\\b\\0\end{pmatrix} = \gamma_0\begin{pmatrix}a\\0\\b\\0\end{pmatrix} = \begin{pmatrix}a\\0\\-b\\0\end{pmatrix} \Rightarrow b=0$

$$\Rightarrow \langle 0|\psi(0)|1/2;(m,\vec{0})\rangle = a\sqrt{2m}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = a\,u^{(1)}_{\vec{0}}$$

Conventionally, the constant a is defined to be $\sqrt{Z_2}$

$$\Rightarrow \psi' = \frac{1}{\sqrt{Z_2}}\psi$$

$$\langle 0|\psi'(0)|1/2, r, (m,\vec{0})\rangle = u^{(r)}_{\vec{0}} \Rightarrow \langle 0|\psi'(x)|r\, p\rangle = e^{-ipx}u^{(r)}_{\vec{p}}$$

The latter is obtained by applying a translation and Lorentz transformations.

This is essentially all one needs to derive LSZ for fermions, which is left as an exercise.

Counterterms and renormalization conditions.

(I refrain from using the ' on renormalized fields)

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}\mu^2\phi^2 + \bar{\psi}(i\not{\partial}-m)\psi - g\bar{\psi}i\gamma_5\psi\phi$$

$$+\frac{1}{2}A\,\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}B\phi^2 + C\bar{\psi}(i\not{\partial})\psi - D\bar{\psi}\psi - E\bar{\psi}i\gamma_5\psi\phi$$

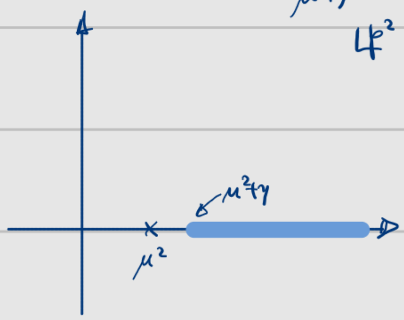
How do we determine these counterterms? We need renormalization conditions analogously to how we proceeded in the case of the all-scalar model 3.

2-point function of the meson:

$$\int d^4x\, d^4y\, e^{-ip'x}e^{-ipy}\langle 0|T\phi(x)\phi(y)|0\rangle \equiv (2\pi)^4\delta^4(p+p')\tilde{D}_r(p^2)$$

We then derived the spectral representation for $\tilde{D}_r(p^2)$:

$$\tilde{D}_r(p^2) = \frac{i}{p^2-\mu^2+i\varepsilon} + \int_{\mu^2+\eta}^{\infty} ds\,\sigma(s)\,\frac{i}{p^2-s+i\varepsilon}$$



$\equiv -i\tilde{\Pi}_r(p^2)$

$$\tilde{D}_r(p^2) = \tilde{D} + \tilde{D}\left[-i\tilde{\Pi}_r\right]\tilde{D} + \dots = \frac{i}{p^2-\mu^2-\tilde{\Pi}_r(p^2)+i\varepsilon}$$

Renormalization conditions: $\tilde{\Pi}_r(\mu^2)=0 \Rightarrow \mu^2$ is the position of the pole (= physical mass)

$\tilde{\Pi}'_r(p^2)\big|_{p^2=\mu^2}=0 \Rightarrow$ the residue of the pole is i

In the case of the nucleon as spin-1/2 fermion, the logic is similar but the details more complicated:

$$\int d^4x\, d^4y\; e^{-ip'x}e^{-ipy}\langle 0|T\,\psi_r(x)\bar{\psi}_r(y)|0\rangle \equiv (2\pi)^4\delta^4(p+p')\tilde{S}_r(p)$$

where $\tilde{S}_r$ is a 4×4 matrix, just like the propagator we discussed and derived for a free theory $\tilde{S}_F(p)=\frac{i}{\not{p}-m+i\varepsilon}$.

What can we say about $\tilde{S}_r(p)$? First of all we can write the most general 4×4 matrix in terms of γ-matrices, which constitute a basis. Moreover, we know that it can only depend on the momentum p and has to be Lorentz invariant:

$$\tilde{S}_r(p)=a(p^2)+b(p^2)\gamma_5+c(p^2)\not{p}+d(p^2)\gamma_5\not{p}+e(p^2)\sigma^{\mu\nu}p_\mu p_\nu$$

The last term is obviously zero. The two terms with γ_5 must be zero if the theory is parity invariant as in our case.

$U_P^{+}\psi U_P=\beta\psi$

$\bar{\psi}=\psi^{+}\gamma^0 \to \bar{\psi}\xrightarrow{P}U_P^{+}\psi^{+}U_P\gamma^0=U_P^{+}\bar{\psi}U_P$

$\psi^{+}\xrightarrow{P}U_P^{+}\psi^{+}U_P$

$\quad\quad\quad\quad \parallel$

$\psi^{+}\beta^{+} \to \bar{\psi}\to\psi^{+}\beta^{+}\gamma_0=\bar{\psi}\beta$

$\tilde{S}_r(p)\xrightarrow{P}\beta\tilde{S}_r\beta$

$a(p^2)$ and $c(p^2)$ are invariant but the $b(p^2)$ and $d(p^2)$ terms flip sign.

So we conclude that

$$\tilde{S}_r(p) = a(p^2) + c(p^2)\not{p}$$

Notice that since $\not{p}^2 = p^2$, we can consider $\tilde{S}_r$ as a function of $\not{p}$, i.e. a function of a matrix, which can be defined as a function of a variable z: $\tilde{S}_r(z) = a(z^2) + c(z^2)z$, and considered as a function of the matrix $\not{p}$ for the case of our interest.

$$\tilde{S}_r(\not{p}) = a(\not{p}^2) + c(p^2)\not{p}$$

Spectral representation.

Going through the same steps that led to the spectral representation for the scalar two-point function we obtain an analogous one for the fermion two-point function. This looks like this:

$$\tilde{S}_r(\not{p}) = \frac{i}{\not{p} - m + i\varepsilon} + i\int_{(m+\mu)^2}^{\infty} ds\, \sigma_+(s)\, \frac{\not{p} + \sqrt{s}}{p^2 - s + i\varepsilon} + i\int_{(m+\mu)^2}^{\infty} ds\, \sigma_-(s)\, \frac{\not{p} - \sqrt{s}}{p^2 - s + i\varepsilon}$$

The reason why the second term appears is that the field $\bar{\psi}$ can create both $J^P = 1/2^+$ as well as $1/2^-$ states. The latter are generated by the lower two components of the Dirac fermion, which are parity eigenstates with eigenvalue -1:

$$(2\pi)^3 \sum_{n, J^P = 1/2^+}' \delta^4(p - p_n) \langle 0|\psi(0)|n\rangle\langle n|\bar{\psi}(0)|0\rangle = \sigma_+(p^2)\,\theta(p^0)\,(\not{p} + m)$$

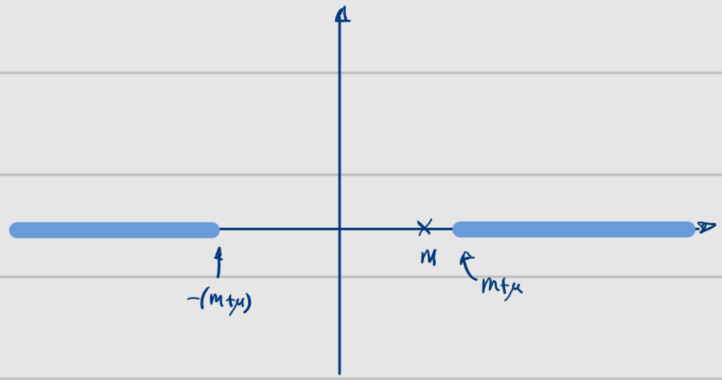
$$(2\pi)^3 \sum_{n, J^P = 1/2^-}' \delta^4(p - p_n) \langle 0|\psi(0)|n\rangle\langle n|\bar{\psi}(0)|0\rangle = \sigma_-(p^2)\,\theta(p^0)\,(\not{p} - m)$$

[$\frac{m \pm \not{p}}{2m}$ are projection operators: $\left(\frac{m \pm \not{p}}{2m}\right)^2 = \frac{m^2 \pm 2m\not{p} + \not{p}^2}{4m^2} = \frac{2m(m \pm \not{p})}{4m^2} = \frac{m \pm \not{p}}{2m}$ ✓]

If we write $\tilde{S}_r(\not{p})$ in the following form:

$$\tilde{S}_r(\not{p}) = \frac{i}{\not{p} - m + i\varepsilon} + \int_{(m+\mu)^2}^{\infty} ds\, \sigma_+(s) \frac{i}{\not{p} - \sqrt{s} + i\varepsilon} + \int_{(m+\mu)^2}^{\infty} ds\, \sigma_-(s) \frac{i}{\not{p} + \sqrt{s} - i\varepsilon}$$

and look at the analytic properties of $\tilde{S}_r(z)$ we see that it has two branch cuts, starting at $z = \pm(m+\mu)$:



Nucleon self-energy.

We continue to follow the same steps as in the scalar case, and now look at 1PI diagrams contributing to $\tilde{S}_r$.

$$\longrightarrow \text{(1PI)} \longrightarrow \; \equiv -i\tilde{\Sigma}(\not{p})$$

which, if one resums the geometric series gives:

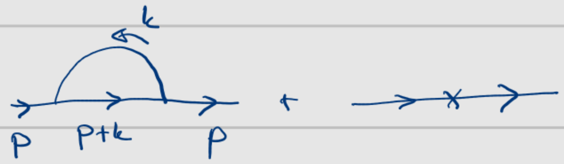
$$\tilde{S}_r(\not{p}) = \frac{i}{\not{p} - m - \tilde{\Sigma}(\not{p}) + i\varepsilon}$$

The conditions that the pole is at $\not{p} = m$ and that the residue is i can be conveniently expressed in terms of $\tilde{\Sigma}(\not{p})$:

$$\tilde{\Sigma}(m) = 0 \quad ; \quad \frac{d\tilde{\Sigma}(\not{p})}{d\not{p}}\bigg|_{\not{p}=m} = 0$$

With these conditions we can fix the counterterms C and D.

Explicit calculation of $\tilde{\Sigma}(\not p)$ to order g^2 -

$-i\tilde{\Sigma}(\not p) =$ 

$$-i\tilde{\Sigma}(\not p) = -i\tilde{\Sigma}^f(\not p) + iC_2\not p - iD_2$$

$$-i\tilde{\Sigma}^f(\not p) = (-ig)^2\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2-\mu^2+i\varepsilon}\, i\gamma_5\, \frac{i(\not p+\not k+m)}{(p+k)^2-m^2+i\varepsilon}\, i\gamma_5$$

$$= -g^2\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2-\mu^2+i\varepsilon}\cdot\frac{(m-\not p-\not k)}{(p+k)^2-m^2+i\varepsilon}$$

Feynman trick

$$\tilde{\Sigma}^f(\not p) = -\frac{ig^2}{(2\pi)^4}\int d^4k\int_0^1 dx\,\frac{m-\not p-\not k}{[k^2+2k\cdot p x+p^2x-m^2x-\mu^2(1-x)+i\varepsilon]^2}$$

$k = k' - px$

$$= \frac{-ig^2}{(2\pi)^4}\int d^4k\int_0^1 dx\,\frac{m-\cancelto{0}{\not k}-\not p(1-x)}{[k^2+p^2x(1-x)-m^2x-\mu^2(1-x)+i\varepsilon]^2}$$

By imposing $\tilde{\Sigma}(m)=0$ and $\left.\frac{d\tilde{\Sigma}(z)}{dz}\right|_{z=\not p=m}=0$ we can fix C_2 and D_2 - In particular we have:

$$\tilde{\Sigma}^f(m) = C_2 m - D_2$$

$$\left.\frac{d\tilde{\Sigma}^f(z)}{dz}\right|_{z=\not p=m} = C_2$$

$$\tilde{\Sigma}^f(m) = \frac{-ig^2}{(2\pi)^4}\int d^4k \int_0^1 dx \, \frac{mx}{\left[k^2 - m^2x^2 - \mu^2(1-x) + i\varepsilon\right]^2}$$

$$\left.\frac{d\tilde{\Sigma}^f}{d\not z}\right|_{\not z=\not p=m} = \frac{-ig^2}{(2\pi)^4}\int d^4k \int_0^1 dx \left\{ \frac{(x-1)}{\left[k^2-m^2x^2-\mu^2(1-x)+i\varepsilon\right]^2} - \frac{4m^2x^2(1-x)}{\left[k^2-m^2x^2-\mu^2(1-x)+i\varepsilon\right]^3} \right\}$$

$$= \frac{-ig^2}{(2\pi)^4}\int d^4k \int_0^1 dx \, \frac{x-1}{\left[k^2-m^2x^2-\mu^2(1-x)+i\varepsilon\right]^3}\left[k^2 - \cancel{m^2x^2} - \mu^2(1-x) + \overset{3}{\cancel{4}}m^2x^2\right]$$

$$= \frac{-ig^2}{(2\pi)^4}\int d^4k \int_0^1 dx \, \frac{(x-1)\left[k^2+3m^2x^2-\mu^2(1-x)\right]}{\left[k^2-m^2x^2-\mu^2(1-x)+i\varepsilon\right]^3}$$

Both are logarithmically divergent.

Self-energy of the meson:

$$\tilde{\Pi}^f(p^2) \sim \int d^4k \, Tr\left[i\gamma_5 \frac{1}{\not p + \not k - m} i\gamma_5 \frac{1}{\not k - m}\right] =$$

$$= \int d^4k \, \frac{4(k^2 + p\cdot k - m^2)}{\left[(p+k)^2 - m^2\right]\left[k^2 - m^2\right]}$$

$$k^2 + p\cdot k - m^2 = \frac{1}{2}\left[(p+k)^2 - m^2 + k^2 - m^2 - m^2\right]$$

$$\tilde{\Pi}^f(p^2) \sim 4\left[2A(m^2) - m^2 \int d^4k \, \frac{1}{\left[(p+k)^2-m^2\right]\left[k^2-m^2\right]}\right]$$