Our fermion-boson Lagrangian with Yukawa coupling:

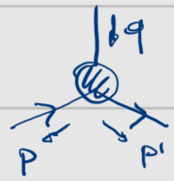
$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\mu^2\phi^2 + \bar{\Psi}(i\partial\!\!\!/ - m)\psi - g\bar{\Psi}i\gamma_5\psi\phi$$
$$+\frac{1}{2}A\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}B\phi^2 + C\bar{\Psi}i\partial\!\!\!/\psi - D\bar{\Psi}\psi - E\bar{\Psi}i\gamma_5\psi\phi$$

we have discussed how to determine $A, \dots, D$ counterterms and now need to determine E.

(In the all-scalar version of this model we had a shift in the names of the CT, because we had one more:

$$\mathcal{L}_{CT}: A\phi + \frac{1}{2}B\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}C\phi^2 + D\partial_\mu\psi^*\partial^\mu\psi - E\psi^*\psi - F\psi^*\psi\phi \quad)$$

The way we fixed F was to look at the vertex

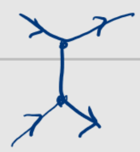


$= -i\tilde{\Gamma}(p^2, p'^2, q^2) \qquad ; \quad q = p + p'$

$=$

$-ig \qquad\qquad -i\tilde{\Gamma}_f(p^2, p'^2, q^2) \qquad -iF$

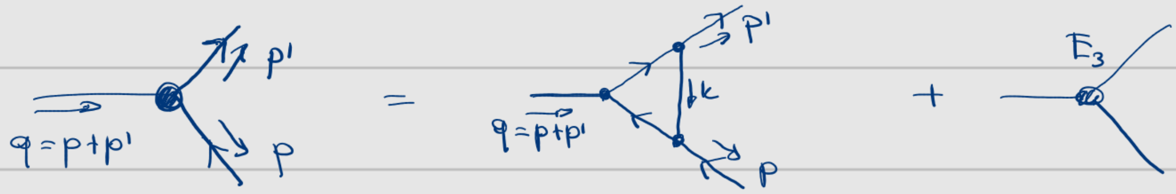
and we then imposed the condition $\tilde{\Gamma}(m^2, m^2, \mu^2) = g$ or $\tilde{\Gamma}_f(m^2, m^2, \mu^2) + F = 0$. The condition is at an unphysical kinematic point, which can be reached by analytic continuation in a process like $NN \to NN$. The diagram



contributes a pole at $t = \mu^2$. Our condition fixes the residue of the pole.

We would like to do the same for our fermionic Yukawa-like theory, but we face an additional difficulty right away because our vertex is sandwiched between two spinors and is in principle a 4×4 matrix:

$$= -i\Gamma(p,p',q) = g\gamma_5 + O(g^3)$$



$$- \bar u(p')\,\gamma_5 g \int \frac{d^4k}{(2\pi)^4} \frac{i}{\not p'+\not k - m + i\varepsilon}\, \gamma_5 g \frac{i}{\not k - \not p - m + i\varepsilon} \cdot \gamma_5 g \frac{i}{k^2-\mu^2+i\varepsilon} \cdot v(p)$$

$$= i g^3\, \bar u(p') \int \frac{d^4k}{(2\pi)^4} \frac{\gamma_5(\not p'+\not k+m)\gamma_5(\not k-\not p+m)\gamma_5}{\underbrace{[(p'+k)^2-m^2]}_{D_1}\underbrace{[(k-p)^2-m^2]}_{D_2}\underbrace{[k^2-\mu^2+i\varepsilon]}_{D_3}} \cdot v(p)$$

$$= i g^3\, \bar u(p')\,\gamma_5 \underbrace{\int \frac{d^4k}{(2\pi)^4}\frac{(\not p'+\not k+m)(\not p-\not k+m)}{D_1 D_2 D_3}}_{I} \cdot v(p)$$

$$I = (\not p'+m)(\not p+m)C_0 + \left[\gamma^\mu(\not p+m) - (\not p'+m)\gamma^\mu\right]C_\mu - \gamma^\mu\gamma^\nu C_{\mu\nu}$$

$$= (\not p'+m)(\not p+m)C_0 + \left[\gamma^\mu \not p - \not p'\gamma^\mu\right]C_\mu - g^{\mu\nu}C_{\mu\nu}$$

where

$$C_{0,\mu,\mu\nu} = \int \frac{d^4k}{(2\pi)^4}\frac{(1,\, k_\mu,\, k_\mu k_\nu)}{D_1 D_2 D_3}$$

For $p^2 = p'^2 = m^2$, the C_0 integral is only a function of q^2.

$$C_\mu = p_\mu C_1 + p'_\mu C_2$$

$$= \gamma^\mu\gamma^\nu\left[(p_\mu - p'_\mu)p_\nu C_1 + p'_\mu(p_\nu - p'_\nu)C_2\right] = \gamma^\mu\gamma^\nu\left[p_\mu p_\nu C_1 - p'_\mu p'_\nu C_2 - p'_\mu p_\nu (C_1 - C_2)\right]$$

$$= m^2(C_1-C_2) - \not{p}'\not{p}(C_1-C_2) = (m^2-\not{p}'\not{p})(C_1-C_2)$$

Since this is sandwiched between $\bar{u}(p')$ and $v(p)$ we can use:

$$(\not{p}+m)\,v(p) = 0 = \bar{u}(p')(\not{p}'-m)$$

to write:

$$\bar{u}(p')\gamma_5\,(m^2-\not{p}'\not{p})\,v(p) = \bar{u}(p')\,(m^2\gamma_5+\not{p}'\gamma_5\not{p})\,v(p)$$

$$= \bar{u}(p')\left(m^2\gamma_5 - m^2\gamma_5\right)v(p) = 0$$

Let us now look closely at $g^{\mu\nu}C_{\mu\nu}$:

$$g^{\mu\nu}C_{\mu\nu} = \int\frac{d^4k}{(2\pi)^4}\,\frac{k^2}{D_1D_2D_3} = \int\frac{d^4k}{(2\pi)^4}\,\frac{D_3+\mu^2}{D_1D_2D_3} = \int\frac{d^4k}{(2\pi)^4}\,\frac{1}{D_1D_2} + \mu^2 C_0$$

$$= B_0 + \mu^2 C_0$$

So, all in all we have:

$$I = (\not{p}'+m)(\not{p}+m)C_0 - B_0 - \mu^2 C_0$$

But if we sandwich the first term between $\bar{u}(p')\gamma_5$ and $v(p)$ we obtain zero, because of the EoM, and the final result is:

$$\bar{u}(p')\gamma_5\, I\, v(p) = -\bar{u}(p')\gamma_5\, v(p)\,\left(B_0+\mu^2 C_0\right)$$

We conclude that, even though the vertex loop correction has the form of a complicated Dirac matrix Γ, it can be reduced to the form of the vertex we have in the original Lagrangian, namely just a γ_5.

Coleman offers a nice argument showing that this result can be derived without doing a direct calculation, let us go through it.

1. Instead of looking at $-i\tilde{\Gamma}(p',p,q)$ we can consider

$$(\not{p}'+m)\,\tilde{\Gamma}(p',p,q)\,(\not{p}-m)\Big|_{p^2=p'^2=m^2}$$

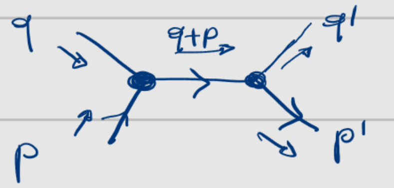
The reason is the following: if $\tilde{\Gamma}(p',p,q)$ is sandwiched between two on-shell spinors we have $\bar{u}(p')\dfrac{(\not{p}'+m)}{2m} = \bar{u}(p')$

and $-\dfrac{(\not{p}-m)}{2m}v(p) = v(p)$

so we haven't changed the matrix element at all.

If one of the fermion legs coming out of the vertex is an internal line, then the vertex will be multiplied by a fermion propagator. For example if I am looking at the

$\phi N \to \phi N$ process



$\sim$

$$\bar{u}(p')\left[-i\tilde{\Gamma}(p',q+p,q')\right]\tilde{S}(q+p)\left[-i\Gamma(q+p,p,q)\right]u(p)$$

$$\tilde{S}(q+p) = \frac{i}{\not{p}+\not{q}-m} = \frac{i(\not{p}+\not{q}+m)}{s-m^2}$$

So, the numerator is exactly what we have multiplied Γ with, and for the residue, this needs to be evaluated at $(p+q)^2 = m^2$.

$\Rightarrow$ we do not lose any generality if we consider

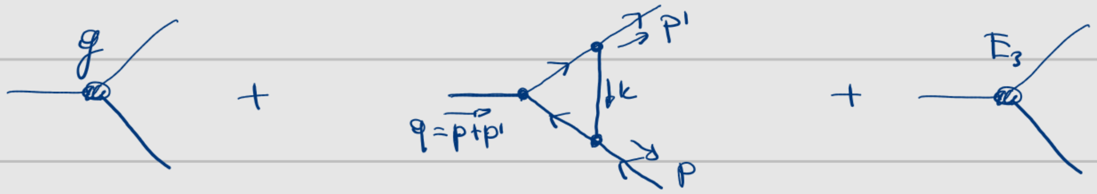
$$(\not{p}'+m)\,\tilde{\Gamma}(p',p,q)\,(\not{p}-m) \,.$$

2. $$(\not{p}'+m)\,\tilde{\Gamma}(p',p,q)\,(\not{p}-m)\Big|_{p^2=p'^2=m^2} = (\not{p}'+m)\, i\gamma_5\,(\not{p}-m)\cdot G(q^2)$$

This we have essentially demonstrated above by a brute force calculation: all possible terms which have appeared in our calculation have been reduced to $i\gamma_5\, G(q^2)$.

Coleman reaches the same conclusion by arguing that if we put the external legs on-shell, the vertex represents the amplitude for creating two nucleons from a pseudoscalar field. The latter has $J^P = 0^-$, and the nucleons must be created in this state $\Rightarrow$ $i\gamma_5$.

The important point is that, once we have shown that the corrected vertex has the same form as the vertex in the Lagrangian, we can impose the renormalization condition exactly as we did in the case of the all-scalar theory:

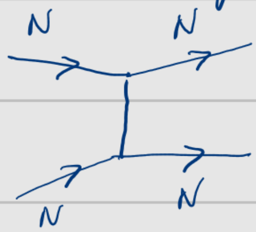


$$i\gamma_5\Big[\, g \;+\; g^3 G(q^2) \;+ g^3 e_3 \;+ O(g^5)\Big]\Big|_{q^2=\mu^2}$$

$= i\gamma_5 g$

$$\Rightarrow \quad G(\mu^2) + e_3 = 0$$

While not quite a measurable quantity, this can be related by analytic continuation to the residue of a pole at $t=\mu^2$ in the nucleon-nucleon scattering amplitude:



$$\sim \quad \frac{g^2}{t-\mu^2} + (\text{terms regular in } t=\mu^2)$$

Note that the constant e_3 is divergent because

$$G(\mu^2) \simeq B_0(\mu^2) + \text{convergent integrals}$$

with $B_0(\mu^2)$ divergent.