Series 1

I. Dimensional analysis

1. *Work out the engineering dimension of the free fermion in 4d obeying*

$$\mathcal{L} = i\bar{\psi}\gamma_\mu \partial^\mu \psi. \tag{1}$$

We know that $[\mathcal{L}] = 4$ and that $[\partial_\mu] = 1$ and also that the gamma matrices are dimensionless so we can work out the following :

$$[\mathcal{L}] = [\partial_\mu] + 2[\psi] \Rightarrow 2[\psi] + 1 = 4 \Rightarrow [\psi] = \frac{3}{2}. \tag{2}$$

2. *Work out the engineering dimension of the real scalar field a in 3d with kinetic term $\frac{1}{2}\partial_\mu a \partial^\mu a$.*

In 3d we know that $[\mathcal{L}]$=3, and also $[\partial_\mu] = 1$ in any number of dimensions, so we can work out that:

$$[\mathcal{L}] = 2\Big([\partial_\mu] + [a]\Big) \Rightarrow 2[a] + 2 = 3 \Rightarrow [a] = \frac{1}{2}. \tag{3}$$

3. *Based on the engineering dimension of a, what kind of potential term can we add to the Lagrangian if we use only dimensionless couplings?*

Based on the engineering dimension of a, and the scaling dimension of the Lagrangian, the potential should scale as $[V] = 3$, so we can add a term of the following form:

$$V = \lambda a^6, \tag{4}$$

where λ is a dimensionless parameter.

4. *Consider a complex scalar field of the form $\phi = ae^{i\chi}$ in 3d. Its Lagrangian contains (among others) of the kinetic term for a and the potential term determined above. What must be the dimension of χ?*

The dimension of χ is zero since χ is a pure phase, i.e. $[\chi] = 0$.

5. *What other terms made of a and χ can be added to the Lagrangian (again without using dimensionful couplings and up tp two derivatives in the fields)?*

We should add some for of kinetic term for χ to the Lagrangian, $\propto f(a)\partial_\mu \chi \partial^\mu \chi$. Dimensional analysis tells us that it must be

$$\propto a^2 \partial_\mu \chi \partial^\mu \chi. \tag{5}$$

When working in curved space, there is another dimensionful we can use to build terms with, namely the Ricci scalar (which has dimension two as it contains two derivatives). In this case, we can also add the term $b\mathcal{R}a^2$, where b is a dimensionless constant.

II. Transformations from the inversion

1. *Perform explicitly an inversion*

$$x^\mu \to \frac{x^\mu}{x^2} \tag{6}$$

followed by a translation by a^μ followed by another inversion. What is the resulting transformation?

We start with an inversion,

$$x'^\mu = \frac{x^\mu}{x^2}. \tag{7}$$

Then we perform a translation, which reads:

$$x''^\mu = x'^\mu - a^\mu. \tag{8}$$

Finally we perform another inversion, such that:

$$x^{*\mu} = \frac{(x'')^\mu}{(x'')^2}. \tag{9}$$

Then insert equation (7) and equation (8) into equation (9) to get:

$$\begin{aligned} x^{*\mu} &= \frac{x'^\mu - a^\mu}{(x'')^2} \\ &= \frac{\frac{x^\mu}{x^2} - a^\mu}{(x'')^2} \\ &= \frac{x^\mu - a^\mu x^2}{x^2 (x'')^2} \\ &= \frac{x^\mu - a^\mu x^2}{x^2 \left(\frac{x^\mu}{x^2} - a^\mu\right)\left(\frac{x_\mu}{x^2} - a_\mu\right)} \\ &= \frac{x^\mu - a^\mu x^2}{x^2 \left(\frac{x^\mu x_\mu}{x^4} - \frac{x^\mu a_\mu}{x^2} - \frac{x_\mu a^\mu}{x^2} + a^\mu a_\mu\right)} \\ &= \frac{x^\mu - a^\mu x^2}{1 - 2ax + a^2 x^2}, \end{aligned} \tag{10}$$

where we used that $x^2 = x^\mu x_\mu$, and also that $a^\mu x_\mu = a_\mu x^\mu = ax$ and that $a^\mu a_\mu = a^2$. This is the special conformal transformation we have seen in the lecture.

2. *Perform explicitly an inversion followed by a dilatation/scale transformation followed by another inversion. What is the resulting transformation?*

We start with an inversion

$$x'^{\mu} = \frac{x^{\mu}}{x^2}. \tag{11}$$

Then we perform a dilatation, which reads

$$x''^{\mu} = e^a x'^{\mu}, \qquad a \in \mathcal{R} \tag{12}$$

Finally we perform another inversion, such that

$$x^{*\mu} = \frac{(x'')^{\mu}}{(x'')^2}. \tag{13}$$

Then insert equation (11) and equation (12) into equation (13) to get:

$$\begin{aligned} x^{*\mu} &= \frac{e^a x'^{\mu}}{\left(x''\right)^2} \\ &= \frac{\frac{e^a x^{\mu}}{x^2}}{\left(x''\right)^2} = \frac{e^a x^{\mu}}{x^2 \left(x''\right)^{\mu} \left(x''\right)_{\mu}} \\ &= \frac{e^a x^{\mu}}{e^{2a} x^2 \left(x'\right)^{\mu} \left(x'\right)_{\mu}} \\ &= \frac{e^{-a} x^{\mu}}{x^2 \left(\frac{x^{\mu}}{x^2}\right)\left(\frac{x_{\mu}}{x^2}\right)} \\ &= \frac{e^{-a} x^{\mu}}{\frac{x^2 x^{\mu} x_{\mu}}{x^4}} \\ &= e^{-a} x^{\mu} \end{aligned} \tag{14}$$

where we used that $x^2 = x^{\mu} x_{\mu}$, and also that $a \in \mathcal{R}$. This is still a dilatation/scale transformation.