Series 2

I. Infinitesimal generators

1. *Work out the infinitesimal generator starting from the finite form of the scale transformation,*

$$x'^{\mu} = \alpha x^{\mu}. \tag{1}$$

We start by parameterizing $\alpha = e^{a},\ a \in \mathcal{R}$, so that :

$$x'^{\mu} = e^{a} x^{\mu}. \tag{2}$$

Then we Taylor expand to first order, i.e. $\mathcal{O}(a^2)$, to obtain the infinitesimal transformation:

$$x'^{\mu} = x^{\mu} + a x^{\mu}. \tag{3}$$

Then we find the generator as

$$G_a = -i\left(\frac{\delta x^{\mu}}{\delta \epsilon_a}\partial_{\mu}\right). \tag{4}$$

In which case we get $\frac{\delta x^{\mu}}{\delta a} = x^{\mu}$, so that

$$G_a = -i x^{\mu}\partial_{\mu} \equiv D \tag{5}$$

which is the generator of dilatation/scale transformations.

2. *Work out the infinitesimal generator starting from the finite form of the SCT,*

$$x'^{\mu} = \frac{x^{\mu} - b^{\mu}\vec{x}^2}{1 - 2\vec{b}\cdot\vec{x} + b^2\vec{x}^2}. \tag{6}$$

We start by writing this as :

$$x'^{\mu} = \left(x^{\mu} - b^{\mu}x^2\right)\frac{1}{1 - \left(2bx - b^2x^2\right)}. \tag{7}$$

This will allows us to use the identity $\frac{1}{1-c} = 1 + c,\ |c| \ll 1$. Using this, we get:

$$\begin{aligned} x'^{\mu} &= \left(x^{\mu} - b^{\mu}x^2\right)\left(1 + 2bx - b^2x^2\right) + O(b^3) \\ &= \left(x^{\mu} - b^{\mu}x^2\right)\left(1 + 2bx\right) + O(b^2) \\ &= x^{\mu} + 2(bx)x^{\mu} - b^{\mu}x^2 + O(b^2). \end{aligned} \tag{8}$$

This is the infinitesimal transformation of the SCT. The same result can be obtained by Taylor expansion. Then, we can use again eq. (4), and the fact that $\frac{\delta x^\mu}{\delta b^\nu} = 2x_\mu \delta^\mu{}_\nu x^\mu$, $\frac{\delta x^\mu}{\delta b^\nu} = -x^2 \delta^\mu{}_\nu$ so that :

$$\begin{aligned} G_a &= -i\Big(2x_\mu \delta^\mu{}_\nu x^\mu \partial_\mu - x^2 \delta^\mu{}_\nu \partial_\mu\Big) \\ &= -i\Big(2x_\nu x^\mu \partial_\mu - x^2 \partial_\nu\Big). \end{aligned} \tag{9}$$

Since these are dummy indices, we can exchange $\nu \iff \mu$ to get:

$$K_\mu = -i\Big(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu\Big). \tag{10}$$

II. Scale factor of the SCT

Work out explicitly the scale factor $\Lambda(x)$ of the SCT given in eq. (6).

We have that

$$\Lambda(x)\eta_{\mu\nu} = \eta_{\rho\sigma} \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} \tag{11}$$

and the desired transformation is

$$x'^\mu = \frac{x^\mu - b^\mu x^2}{1 - 2bx + b^2 x^2}, \tag{12}$$

which can be rewritten as

$$x^\mu = x'^\mu \Big(1 - 2bx + b^2 x^2\Big) + b^\mu x^2. \tag{13}$$

Then, it is easy to derive that

$$\frac{\partial x^\rho}{\partial x'^\mu} = \delta^\rho{}_\mu \Big(1 - 2bx + b^2 x^2\Big). \tag{14}$$

And similarly for sigma:

$$\frac{\partial x^\sigma}{\partial x'^\nu} = \delta^\sigma{}_\nu \Big(1 - 2bx + b^2 x^2\Big). \tag{15}$$

Then we should plug eq. (14) and eq. (15) back into eq. (11) to get

$$\begin{aligned} \Lambda(x)\eta_{\mu\nu} &= \eta_{\rho\sigma} \delta^\rho{}_\mu \delta^\sigma{}_\nu \Big(1 - 2bx + b^2 x^2\Big)\Big(1 - 2bx + b^2 x^2\Big) \\ &= \eta_{\mu\nu} \Big(1 - 2bx + b^2 x^2\Big)^2. \end{aligned} \tag{16}$$

And hence, the scale factor for the SCT is

$$\Lambda(x) = \Big(1 - 2bx + b^2 x^2\Big)^2. \tag{17}$$

III. Commutation rules of the conformal algebra.

Verify explicitly the commutation rules of the generators of the conformal algebra involving the new generators D and K_μ as given in the lecture.

1.

$$\begin{aligned}[D, P_\mu] &= [-ix^\nu\partial_\nu, -i\partial_\mu] \\ &= i^2 x^\nu \partial_\nu \partial_\mu - i^2 \partial_\mu x^\nu \partial_\nu \\ &= 0 - i^2 \delta_\mu^{\ \nu}\partial_\nu \\ &= -i^2\partial_\mu \\ &= i\Big(-i\partial_\mu\Big) = iP_\mu\end{aligned} \tag{18}$$

2.

$$\begin{aligned}[D, K_\mu] &= [-ix^\rho\partial_\rho, -2ix_\mu x^\nu\partial_\nu + ix^2\partial_\mu] \\ &= [-ix^\rho\partial_\rho, -2ix_\mu x^\nu\partial_\nu] + [-ix^\rho\partial_\rho, ix^2\partial_\mu] \\ &= \Big(2i^2 x^\rho\partial_\rho(x_\mu x^\nu)\partial_\nu - 2i^2 x_\mu x^\nu \partial_\nu x^\rho \partial_\rho\Big) + \Big(-i^2 x^\rho \partial_\rho (x_\sigma x^\sigma)\partial_\mu - (-i^2)x^2\partial_\mu x^\rho \partial_\rho\Big) \\ &= \Big(-2x^\rho\eta_{\rho\mu}x^\nu\partial_\nu - 2x^\rho x_\mu \delta_\rho^{\ \nu}\partial_\nu + 2x_\mu x^\nu \delta_\nu^{\ \rho}\partial_\rho\Big) \\ &+ \Big(x^\rho\eta_{\rho\sigma}x^\sigma\partial_\mu + x^\rho x_\sigma\delta_\rho^{\ \sigma}\partial_\mu - x^2\delta_\mu^{\ \rho}\partial_\rho\Big) \\ &= \Big(-2x_\mu x^\nu\partial_\nu - 2x_\mu x^\nu\partial_\nu + 2x_\mu x^\nu\partial_\nu\Big) + \Big(x_\sigma x^\sigma\partial_\mu + x_\rho x^\rho\partial_\mu - x^2\partial_\mu\Big) \\ &= -2x_\mu x^\nu\partial_\nu + x^2\partial_\mu = -iK_\mu.\end{aligned} \tag{19}$$

Since

$$-iK_\mu = -i(-i)(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu) = -2x_\mu x^\nu\partial_\nu + x^2\partial_\mu. \tag{20}$$

3.

$$\begin{aligned}[K_\mu, P_\nu] &= [-2ix_\mu x^\sigma\partial_\sigma + ix^2\partial_\mu, -i\partial_\nu] \\ &= [-2ix_\mu x^\sigma\partial_\sigma, -i\partial_\nu] + [ix^2\partial_\mu, -i\partial_\nu] \\ &= \Big(2i^2 x_\mu x^\sigma\partial_\sigma\partial_\nu - (-2i)(-i)\partial_\nu(x_\mu x^\sigma)\partial_\sigma\Big) + \Big(-i^2x^2\partial_\mu\partial_\nu - (-i^2)\partial_\nu(x^2)\partial_\mu\Big) \\ &= -2i^2\delta_{\nu\mu}x^\sigma\partial_\sigma - 2i^2 x_\mu\delta_\nu^{\ \sigma}\partial_\sigma + i^2\partial_\nu(x_\rho x^\rho)\partial_\mu \\ &= 2\delta_{\nu\mu}x^\sigma\partial_\sigma + 2x_\mu\partial_\nu + i^2\delta_{\nu\rho}x^\rho\partial_\mu + i^2 x_\rho\delta_\nu^{\ \rho}\partial_\mu \\ &= 2\delta_{\nu\mu}x^\sigma\partial_\sigma + 2x_\mu\partial_\nu - x_\nu\partial_\mu - x_\nu\partial_\mu \\ &= 2\delta_{\nu\mu}x^\sigma\partial_\sigma + 2x_\mu\partial_\nu - 2x_\nu\partial_\mu \\ &= 2\Big(\delta_{\mu\nu}x^\sigma\partial_\sigma + x_\mu\partial_\nu - x_\nu\partial_\mu\Big) = 2i\Big(\eta_{\mu\nu}D - L_{\mu\nu}\Big).\end{aligned} \tag{21}$$

Since

$$\begin{aligned} 2i\Big(\eta_{\mu\nu}D - L_{\mu\nu}\Big) &= 2i\eta_{\mu\nu}D - 2iL_{\mu\nu} \\ &= 2i\eta_{\mu\nu}(-ix^{\mu}\partial_{\mu}) - 2i(i)(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu}) \\ &= 2\eta_{\mu\nu}x^{\mu}\partial_{\mu} + 2x_{\mu}\partial_{\nu} - 2x_{\nu}\partial_{\mu}. \end{aligned} \tag{22}$$

4.

$$\begin{aligned} [K_{\mu}, L_{\nu\rho}] &= [-2ix_{\mu}x^{\sigma}\partial_{\sigma} + ix^2\partial_{\mu}, ix_{\nu}\partial_{\rho} - ix_{\rho}\partial_{\nu}] \\ &= [-2ix_{\mu}x^{\sigma}\partial_{\sigma}, ix_{\nu}\partial_{\rho}] + [-2ix_{\mu}x^{\sigma}\partial_{\sigma}, -ix_{\rho}\partial_{\nu}] + [ix^2\partial_{\mu}, ix_{\nu}\partial_{\rho}] + [ix^2\partial_{\mu}, -ix_{\rho}\partial_{\nu}] \\ &= \Big(-2i(i)x_{\mu}x^{\sigma}\partial_{\sigma}x_{\nu}\partial_{\rho} - (-2i)(i)x_{\nu}\partial_{\rho}(x_{\mu}x^{\sigma})\partial_{\sigma}\Big) \\ &+ \Big(-2i(-i)x_{\mu}x^{\sigma}\partial_{\sigma}x_{\rho}\partial_{\nu} - (-2i)(-i)x_{\rho}\partial_{\nu}(x_{\mu}x^{\sigma})\partial_{\sigma}\Big) \\ &+ \Big(i^2x^2\partial_{\mu}x_{\nu}\partial_{\rho} - i^2x_{\nu}\partial_{\rho}(x^2)\partial_{\mu}\Big) \\ &+ \Big(-i^2x^2\partial_{\mu}x_{\rho}\partial_{\nu} - (-i)(i)x_{\rho}\partial_{\nu}(x^2)\partial_{\mu}\Big) \\ &= -2i^2x_{\mu}x^{\sigma}\delta_{\sigma\nu}\partial_{\rho} + 2i^2x_{\nu}\delta_{\rho\mu}x^{\sigma}\partial_{\sigma} + 2i^2x_{\nu}x_{\mu}\delta_{\rho}{}^{\sigma}\partial_{\sigma} \\ &+ 2i^2x_{\mu}x^{\sigma}\delta_{\sigma\rho}\partial_{\nu} - 2i^2x_{\rho}\delta_{\nu\mu}x^{\sigma}\partial_{\sigma} - 2i^2x_{\rho}x_{\mu}\delta_{\nu}{}^{\sigma}\partial_{\sigma} \\ &+ i^2x^2\delta_{\mu\nu}\partial_{\rho} - i^2x_{\nu}\delta_{\rho\sigma}x^{\sigma}\partial_{\mu} - i^2x_{\nu}x_{\sigma}\delta_{\rho}{}^{\sigma}\partial_{\mu} \\ &- i^2x^2\delta_{\mu\rho}\partial_{\nu} + i^2x_{\rho}\delta_{\nu\sigma}x^{\sigma}\partial_{\mu} + i^2x_{\rho}x_{\sigma}\delta_{\nu}{}^{\sigma}\partial_{\mu} \\ &= 2i^2x_{\nu}\delta_{\rho\mu}x^{\sigma}\partial_{\sigma} - 2i^2x_{\rho}\delta_{\nu\mu}x^{\sigma}\partial_{\sigma} + i^2x^2\delta_{\mu\nu}\partial_{\rho} - i^2x^2\delta_{\mu\rho}\partial_{\nu} \\ &= i\Big(2i\delta_{\rho\mu}x_{\nu}x^{\sigma}\partial_{\sigma} - i\delta_{\rho\mu}x^2\partial_{\nu} - 2i\delta_{\mu\nu}x_{\rho}x^{\sigma}\partial_{\sigma} + i\delta_{\mu\nu}x^2\partial_{\rho}\Big) \\ &= i\Big(-i\delta_{\mu\nu}\big(2x_{\rho}x^{\sigma}\partial_{\sigma} - x^2\partial_{\rho}\big) + i\delta_{\rho\mu}\big(2x_{\nu}x_{\sigma}\partial_{\sigma} - x^2\partial_{\nu}\big)\Big) \\ &= i\Big(\delta_{\mu\nu}K_{\rho} - \delta_{\mu\rho}K_{\nu}\Big). \end{aligned} \tag{23}$$