Series 3

I. Representations of the conformal group

1. *Verify the commutation relations given in eq. (2.16) of the notes.*

We want to verify the commutation relations given in the notes. In order to do so, we will use the following:

$$\begin{aligned}\tilde{\Delta} &= -ix^{\mu}\partial_{\mu},\\ \kappa_{\mu} &= -i(2x_{\mu}x^{\nu}\partial_{\nu} - x^2\partial_{\mu}),\\ S_{\mu\nu} &= i(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu}).\end{aligned} \tag{1}$$

Using these, we get:

(a)

$$\begin{aligned}[\tilde{\Delta}, S_{\mu\nu}] &= [-ix^{\rho}\partial_{\rho}, ix_{\mu}\partial_{\nu} - ix_{\nu}\partial_{\mu}]\\ &= [-ix^{\rho}\partial_{\rho}, ix_{\mu}\partial_{\nu}] + [-ix^{\rho}\partial_{\rho}, -ix_{\nu}\partial_{\mu}]\\ &= \Big(-i^2x^{\rho}\partial_{\rho}\big(x_{\mu}\partial_{\nu}\big) + i^2x_{\mu}\partial_{\nu}\big(x^{\rho}\partial_{\rho}\big)\Big) + \Big(i^2x^{\rho}\partial_{\rho}\big(x_{\nu}\partial_{\mu}\big) - i^2x_{\nu}\partial_{\mu}\big(x^{\rho}\partial_{\rho}\big)\Big)\\ &= -i^2x^{\rho}\delta_{\rho\mu}\partial_{\nu} + i^2x_{\mu}\delta_{\nu}{}^{\rho}\partial_{\rho} + i^2x^{\rho}\delta_{\rho\nu}\partial_{\mu} - i^2x_{\nu}\delta_{\mu}{}^{\rho}\partial_{\rho}\\ &= -i^2x_{\mu}\partial_{\nu} + i^2x_{\mu}\partial_{\nu} + i^2x_{\nu}\partial_{\mu} - i^2x_{\nu}\partial_{\mu}\\ &= 0.\end{aligned} \tag{2}$$

(b)

$$\begin{aligned}[\tilde{\Delta}, \kappa_{\mu}] &= [-ix^{\rho}\partial_{\rho}, -2ix_{\mu}x^{\nu}\partial_{\nu} + ix^2\partial_{\mu}]\\ &= [-ix^{\rho}\partial_{\rho}, -2ix_{\mu}x^{\nu}\partial_{\nu}] + [-ix^{\rho}\partial_{\rho}, ix^2\partial_{\mu}]\\ &= \big(2i^2x^{\rho}\partial_{\rho}\big(x_{\mu}x^{\nu}\big)\partial_{\nu} - 2i^2x_{\mu}x^{\nu}\partial_{\nu}x^{\rho}\partial_{\rho}\big) + \big(-i^2x^{\rho}\partial_{\rho}\big(x_{\sigma}x^{\sigma}\big)\partial_{\mu} - (-i^2)x^2\partial_{\mu}x^{\rho}\partial_{\rho}\big)\\ &= \big(-2x^{\rho}\delta_{\rho\mu}x^{\nu}\partial_{\nu} - 2x^{\rho}x_{\mu}\delta_{\rho}{}^{\nu}\partial_{\nu} + 2x_{\mu}x^{\nu}\delta_{\nu}{}^{\rho}\partial_{\rho}\big)\\ &+ \big(x^{\rho}\delta_{\rho\sigma}x^{\sigma}\partial_{\mu} + x^{\rho}x_{\sigma}\delta_{\rho}{}^{\sigma}\partial_{\mu} - x^2\delta_{\mu}{}^{\rho}\partial_{\rho}\big)\\ &= \big(-2x_{\mu}x^{\nu}\partial_{\nu} - 2x_{\mu}x^{\nu}\partial_{\nu} + 2x_{\mu}x^{\nu}\partial_{\nu}\big) + \big(x_{\sigma}x^{\sigma}\partial_{\mu} + x_{\rho}x^{\rho}\partial_{\mu} - x^2\partial_{\mu}\big)\\ &= -2x_{\mu}x^{\nu}\partial_{\nu} + x^2\partial_{\mu}\\ &= -i\kappa_{\mu}.\end{aligned} \tag{3}$$

(c)

$$
\begin{aligned}
[\kappa_\nu, \kappa_\mu] &= [-2ix_\nu x^\rho \partial_\rho + ix^2\partial_\nu, -2ix_\mu x^\sigma \partial_\sigma + ix^2\partial_\mu] \\
&= [-2ix_\nu x^\rho \partial_\rho, -2ix_\mu x^\sigma \partial_\sigma] + [-2ix_\nu x^\rho \partial_\rho, ix^2\partial_\mu] \\
&+ [ix^2\partial_\nu, -2ix_\mu x^\sigma \partial_\sigma] + [ix^2\partial_\nu, ix^2\partial_\mu] \\
&= \Big(4i^2 x_\nu x^\rho \partial_\rho (x_\mu x^\sigma)\partial_\sigma - 4i^2 x_\mu x^\sigma \partial_\sigma (x_\nu x^\rho)\partial_\rho\Big) \\
&+ \Big(-2i^2 x_\nu x^\rho \partial_\rho (x_\omega x^\omega)\partial_\mu + 2i^2 x^2 \partial_\mu (x_\nu x^\rho)\partial_\rho\Big) \\
&+ \Big(-2i^2 x^2 \partial_\nu (x_\mu x^\sigma)\partial_\sigma + 2i^2 x_\mu x^\sigma \partial_\sigma (x_\zeta x^\zeta)\partial_\nu\Big) \\
&+ \Big(i^2 x^2 \partial_\nu (x_\xi x^\xi)\partial_\mu - i^2 x^2 \partial_\mu (x_\gamma x^\gamma)\partial_\nu\Big) \\
&= \Big(4i^2 x_\nu x^\rho \delta_{\rho\mu} x^\sigma \partial_\sigma + 4i^2 x_\nu x^\rho x_\mu \delta_\rho{}^\sigma \partial_\sigma - 4i^2 x_\mu x^\sigma \delta_{\sigma\nu} x^\rho \partial_\rho - 4i^2 x_\mu x^\sigma x_\nu \delta_\sigma{}^\rho \partial_\rho\Big) \\
&+ \Big(-2i^2 x_\nu x^\rho \delta_{\rho\omega} x^\omega \partial_\mu - 2i^2 x_\nu x^\rho x_\omega \delta_\rho{}^\omega \partial_\mu + 2i^2 x^2 \delta_{\mu\nu} x^\rho \partial_\rho + 2i^2 x^2 x_\nu \delta_\mu{}^\rho \partial_\rho\Big) \\
&+ \Big(-2i^2 x^2 \delta_{\nu\mu} x^\sigma \partial_\sigma - 2i^2 x^2 x_\mu \delta_\nu{}^\sigma \partial_\sigma + 2i^2 x_\mu x^\sigma \delta_{\sigma\zeta} x^\zeta \partial_\nu + 2i^2 x_\mu x^\sigma x_\zeta \delta_\sigma{}^\zeta \partial_\nu\Big) \\
&+ \Big(i^2 x^2 x_\xi \delta_\nu{}^\xi \partial_\mu + i^2 x^2 x^\xi \delta_{\xi\nu} \partial_\mu - i^2 x^2 \delta_{\mu\gamma} x^\gamma \partial_\nu - i^2 x^2 x_\gamma \delta_\mu{}^\gamma \partial_\nu\Big) \\
&= -4x_\nu x_\mu x^\sigma \partial_\sigma - 4x_\nu x^\sigma x_\mu \partial_\sigma + 4x_\mu x_\nu x^\rho \partial_\rho + 4x_\mu x^\sigma x_\nu \partial_\sigma + 2x_\nu x^2 \partial_\mu \\
&+ 2x_\nu x^\omega x_\omega \partial_\mu - 2x^2 \delta_{\mu\nu} x^\rho \partial_\rho - 2x^2 x_\nu \partial_\mu + 2x^2 \delta_{\nu\mu} x^\sigma \partial_\sigma + 2x^2 x_\mu \partial_\nu \\
&- 2x_\mu x^\sigma x_\sigma \partial_\nu - 2x_\mu x^\sigma x_\sigma \partial_\nu - x^2 x_\nu \partial_\mu - x^2 x_\nu \partial_\mu + x^2 x_\mu \partial_\nu + x^2 x_\mu \partial_\nu \\
&= 0.
\end{aligned}
\tag{4}
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(d)

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\begin{aligned}
[\kappa_\mu, S_{\nu\rho}] &= [-2ix_\mu x^\sigma \partial_\sigma + ix^2\partial_\mu, ix_\nu\partial_\rho - ix_\rho\partial_\nu] \\
&= [-2ix_\mu x^\sigma\partial_\sigma, ix_\nu\partial_\rho] + [-2ix_\mu x^\sigma\partial_\sigma, -ix_\rho\partial_\nu] + [ix^2\partial_\mu, ix_\nu\partial_\rho] + [ix^2\partial_\mu, -ix_\rho\partial_\nu] \\
&= \Big(-2i(i)x_\mu x^\sigma\partial_\sigma x_\nu\partial_\rho - (-2i)(i)x_\nu\partial_\rho(x_\mu x^\sigma)\partial_\sigma\Big) \\
&+ \Big(-2i(-i)x_\mu x^\sigma\partial_\sigma x_\rho\partial_\nu - (-2i)(-i)x_\rho\partial_\nu(x_\mu x^\sigma)\partial_\sigma\Big) \\
&+ \Big(i^2x^2\partial_\mu x_\nu\partial_\rho - i^2x_\nu\partial_\rho(x^2)\partial_\mu\Big) \\
&+ \Big(-i^2x^2\partial_\mu x_\rho\partial_\nu - (-i)(i)x_\rho\partial_\nu(x^2)\partial_\mu\Big) \\
&= -2i^2x_\mu x^\sigma\delta_{\sigma\nu}\partial_\rho + 2i^2x_\nu\delta_{\rho\mu}x^\sigma\partial_\sigma + 2i^2x_\nu x_\mu\delta_\rho{}^\sigma\partial_\sigma \\
&+ 2i^2x_\mu x^\sigma\delta_{\sigma\rho}\partial_\nu - 2i^2x_\rho\delta_{\nu\mu}x^\sigma\partial_\sigma - 2i^2x_\rho x_\mu\delta_\nu{}^\sigma\partial_\sigma \\
&+ i^2x^2\delta_{\mu\nu}\partial_\rho - i^2x_\nu\delta_{\rho\sigma}x^\sigma\partial_\mu - i^2x_\nu x_\sigma\delta_\rho{}^\sigma\partial_\mu \\
&- i^2x^2\delta_{\mu\rho}\partial_\nu + i^2x_\rho\delta_{\nu\sigma}x^\sigma\partial_\mu + i^2x_\rho x_\sigma\delta_\nu{}^\sigma\partial_\mu \\
&= 2i^2x_\nu\delta_{\rho\mu}x^\sigma\partial_\sigma - 2i^2x_\rho\delta_{\nu\mu}x^\sigma\partial_\sigma + i^2x^2\delta_{\mu\nu}\partial_\rho - i^2x^2\delta_{\mu\rho}\partial_\nu \\
&= i\Big(2i\delta_{\rho\mu}x_\nu x^\sigma\partial_\sigma - i\delta_{\rho\mu}x^2\partial_\nu - 2i\delta_{\mu\nu}x_\rho x^\sigma\partial_\sigma + i\delta_{\mu\nu}x^2\partial_\rho\Big) \\
&= i\Big(-i\delta_{\mu\nu}\big(2x_\rho x^\sigma\partial_\sigma - x^2\partial_\rho\big) + i\delta_{\rho\mu}\big(2x_\nu x_\sigma\partial_\sigma - x^2\partial_\nu\big)\Big) \\
&= i\Big(\delta_{\mu\nu}\kappa_\rho - \delta_{\mu\rho}\kappa_\nu\Big) \\
&= i\Big(\eta_{\mu\nu}\kappa_\rho - \eta_{\mu\rho}\kappa_\nu\Big).
\end{aligned}
\tag{5}
$$

(e)

$$
\begin{aligned}
[S_{\mu\nu}, S_{\rho\sigma}] &= [ix_\mu\partial_\nu - ix_\nu\partial_\mu, ix_\rho\partial_\sigma - ix_\sigma\partial_\rho] \\
&= [ix_\mu\partial_\nu, ix_\rho\partial_\sigma] + [ix_\mu\partial_\nu, -ix_\sigma\partial_\rho] + [-ix_\nu\partial_\mu, ix_\rho\partial_\sigma] + [-ix_\nu\partial_\mu, -ix_\sigma\partial_\rho] \\
&= \Big(i^2x_\mu\partial_\nu(x_\rho)\partial_\sigma - i^2x_\rho\partial_\sigma(x_\mu)\partial_\nu\Big) + \Big(-i^2x_\mu\partial_\nu(x_\sigma)\partial_\rho + i^2x_\sigma\partial_\rho(x_\mu)\partial_\nu\Big) \\
&+ \Big(-i^2x_\nu\partial_\mu(x_\rho)\partial_\sigma + i^2x_\rho\partial_\sigma(x_\nu)\partial_\mu\Big) + \Big(i^2x_\nu\partial_\mu(x_\sigma)\partial_\rho - i^2x_\sigma\partial_\rho(x_\nu)\partial_\mu\Big) \\
&= -x_\mu\delta_{\nu\rho}\partial_\sigma + x_\rho\delta_{\sigma\mu}\partial_\nu + x_\mu\delta_{\nu\sigma}\partial_\rho - x_\sigma\delta_{\rho\mu}\partial_\nu \\
&+ x_\nu\delta_{\mu\rho}\partial_\sigma - x_\rho\delta_{\sigma\nu}\partial_\mu - x_\nu\delta_{\mu\sigma}\partial_\rho + x_\sigma\delta_{\rho\nu}\partial_\mu \\
&= \eta_{\nu\rho}\big(x_\sigma\partial_\mu - x_\mu\partial_\sigma\big) + \eta_{\sigma\mu}\big(x_\rho\partial_\nu - x_\nu\partial_\rho\big) \\
&+ \eta_{\nu\sigma}\big(x_\mu\partial_\rho - x_\rho\partial_\mu\big) + \eta_{\mu\rho}\big(x_\nu\partial_\sigma - x_\sigma\partial_\nu\big) \\
&= i\eta_{\nu\rho}i\big(x_\mu\partial_\sigma - x_\sigma\partial_\mu\big) + \dots \\
&= i\Big(\eta_{\nu\rho}S_{\mu\sigma} + \eta_{\mu\sigma}S_{\nu\rho} - \eta_{\mu\rho}S_{\nu\sigma} - \eta_{\nu\sigma}S_{\mu\rho}\Big).
\end{aligned}
\tag{6}
$$

So we can sum up the algebra as:

$$\begin{aligned} [\tilde{\Delta}, S_{\mu\nu}] &= 0, \\ [\tilde{\Delta}, \kappa_\mu] &= -i\kappa_\mu, \\ [\kappa_\nu, \kappa_\mu] &= 0, \\ [\kappa_\mu, S_{\nu\rho}] &= i\Big(\eta_{\mu\nu}\kappa_\rho - \eta_{\mu\rho}\kappa_\nu\Big), \\ [S_{\mu\nu}, S_{\rho\sigma}] &= i\Big(\eta_{\nu\rho}S_{\mu\sigma} + \eta_{\mu\sigma}S_{\nu\rho} - \eta_{\mu\rho}S_{\nu\sigma} - \eta_{\nu\sigma}S_{\mu\rho}\Big). \end{aligned} \tag{7}$$

2. *Derive (2.17) in the notes using (2.11) and the Hausdorff formula.*

We will use the Hausdorff formula

$$e^{-A}Be^{A} = B + [B, A] + \frac{1}{2!}[[B, A], A] + \dots \tag{8}$$

(a) We will start with the dilatation:

$$e^{ix^\rho P_\rho} D e^{-ix^\rho P_\rho}, \tag{9}$$

where $A = -ix^\rho P_\rho$ and $B = D$. Then, using eq. (8) we get:

$$\begin{aligned} e^{ix^\rho P_\rho} D e^{-ix^\rho P_\rho} &= D + [D, -ix^\rho P_\rho] + \frac{1}{2}[[D, -ix^\rho P_\rho], -x^\sigma P_\sigma] \\ &= D - ix^\rho [D, P_\rho] - \frac{1}{2} ix^\rho [[D, P_\rho], -ix^\sigma P_\sigma] \\ &= D - ix^\rho i P_\rho + \frac{1}{2} i^2 x^\rho x^\sigma [[D, P_\rho], P_\sigma] \\ &= D - ix^\rho i P_\rho - \frac{1}{2} i x^\rho x^\sigma [P_\rho, P_\sigma] \\ &= D + x^\rho P_\rho = D + x^\nu P_\nu. \end{aligned} \tag{10}$$

And we used that $[D, P_\rho] = iP_\rho$ and also that $[P_\mu, P_\nu] = 0$.

(b) For the SCT we have the following:

$$e^{ix^\rho P_\rho} K_\mu e^{-ix^\rho P_\rho}, \tag{11}$$

where $A = -ix^\rho P_\rho$ and $B = K_\mu$. Then, using eq. (8) we derive:

$$\begin{aligned}
e^{ix^\rho P_\rho} K_\mu e^{-ix^\rho P_\rho} &= K_\mu + [K_\mu, -ix^\rho P_\rho] + \frac{1}{2}[[K_\mu, ix^\rho P_\rho], -ix^\sigma P_\sigma] \\
&= K_\mu - ix^\rho [K_\mu, P_\rho] + \frac{1}{2} i^2 x^\rho x^\sigma [K_\mu, P_\rho], P_\sigma] \\
&= K_\mu - ix^\rho \Big(2i\eta_{\mu\rho} D - 2iL_{\mu\rho} \Big) - \frac{1}{2} x^\rho x^\sigma [2i\eta_{\mu\rho} D - 2iL_{\mu\rho}, P_\sigma] \\
&= K_\mu + 2x^\rho \eta_{\mu\rho} D - 2x^\rho L_{\mu\rho} - \frac{1}{2} x^\rho x^\sigma [2i\eta_{\mu\rho} D, P_\sigma] - \frac{1}{2} x^\rho x^\sigma [-2iL_{\mu\rho}, P_\sigma] \\
&= K_\mu + 2x_\mu D - 2x^\rho L_{\mu\rho} - ix^\rho x^\sigma [D, P_\sigma] + ix^\rho x^\sigma [L_{\mu\rho}, P_\sigma] \\
&= K_\mu + 2x_\mu D - 2x^\rho L_{\mu\rho} i x^\rho x^\sigma \eta_{\mu\rho} i P_\sigma - ix^\rho x^\sigma [P_\sigma, L_{\mu\rho}] \\
&= K_\mu + 2x_\mu D - 2x^\rho L_{\mu\rho} + x_\mu x^\sigma P_\sigma - ix^\rho x^\sigma \Big(i\eta_{\sigma\mu} P_\rho - i\eta_{\sigma\rho} P_\mu \Big) \\
&= K_\mu + 2x_\mu D - 2x^\rho L_{\mu\rho} + x_\mu x^\sigma P_\sigma + x^\rho x^\sigma \eta_{\sigma\mu} P_\rho - x^\rho x^\sigma \eta_{\sigma\rho} P_\mu \\
&= K_\mu + 2x_\mu D - 2x^\rho L_{\mu\rho} + 2x_\mu x^\nu P_\nu - x^2 P_\mu \\
&= K_\mu + 2x_\mu D - 2x^\nu L_{\mu\nu} + 2x_\mu x^\nu P_\nu - x^2 P_\mu .
\end{aligned} \tag{12}$$

Here we used that $[D, P_\rho] = iP_\rho$ and that $[P_\rho, L_{\mu\nu}] = i(\eta_{\rho\mu} P_\nu - \eta_{\rho\nu} P_\mu)$, and we also renamed certain dummy indices.

3. *Write down from there the full transformation rules of the fields under D and K_μ (2.18).*

   In order to derive the full transformation rules, we use the above results but we replace $D = \tilde{\Delta}$, $L_{\mu\nu} = S_{\mu\nu}$ and finally $K_\mu = \kappa_\mu$. Then we get the following:

   - $$D\Phi(x) = \tilde{\Delta}\Phi(x) - ix^\nu \partial_\nu \Phi(x) = \Big(-ix^\nu \partial_\nu + \tilde{\Delta} \Big) \Phi(x), \tag{13}$$
   - $$K_\mu \Phi(x) = \{ \kappa_\mu + 2x_\mu \tilde{\Delta} - 2x^\nu S_{\mu\nu} + -2ix_\mu x^\nu \partial_\nu + ix^2 \partial_\mu \} \Phi(x). \tag{14}$$

II. The energy-momentum tensor

1. *Verify that $\partial_\mu \partial_\lambda \partial_\rho X^{\lambda\rho\mu\nu} = 0$ for $X^{\lambda\rho\mu\nu}$ defined as in (2.25) of the notes.*

   We want to verify that $\partial_\mu \partial_\lambda \partial_\rho X^{\lambda\rho\mu\nu} = 0$, where

   $$X^{\lambda\rho\mu\nu} = \frac{2}{d-2} \left(\eta^{\lambda\rho} \sigma_+^{\mu\nu} - \eta^{\lambda\mu} \sigma_+^{\rho\nu} - \eta^{\lambda\nu} \sigma_+^{\mu\rho} + \eta^{\mu\nu} \sigma_+^{\lambda\rho} - \frac{1}{d-1} (\eta^{\lambda\rho} \eta^{\mu\nu} - \eta^{\lambda\mu} \eta^{\rho\nu}) \sigma_{+a}^{a} \right). \tag{15}$$

   We have to calculate the following:

   (a) $\partial_\mu \partial_\lambda \partial_\rho \eta^{\lambda\rho} \sigma_+^{\mu\nu} = \partial_\mu \partial_\lambda \partial^\lambda \sigma_+^{\mu\nu} = \Box \partial_\mu \sigma_+^{\mu\nu} = \Box \sigma_{+,\ \mu}^{\mu\nu}$.

(b) $\partial_\mu\partial_\lambda\partial_\rho\eta^{\lambda\mu}\sigma_+^{\rho\nu} = \partial^\lambda\partial_\lambda\partial_\rho\sigma_+^{\rho\nu} = \Box\partial_\rho\sigma_+^{\rho\nu} = \Box\sigma_{+,\ \rho}^{\rho\nu} = \Box\sigma_{+,\ \mu}^{\mu\nu}$.

(c) $\partial_\mu\partial_\lambda\partial_\rho\eta^{\lambda\nu}\sigma_+^{\mu\rho} = \partial_\mu\partial^\nu\partial_\rho\sigma_+^{\mu\rho} = \partial^\nu\sigma_{+,\ \mu\rho}^{\mu\rho}$.

(d) $\partial_\mu\partial_\lambda\partial_\rho\eta^{\mu\nu}\sigma_+^{\lambda\rho} = \partial_\rho\partial^\nu\partial_\lambda\sigma_+^{\lambda\rho} = \partial^\nu\sigma_{+,\ \lambda\rho}^{\lambda\rho} = .\partial^\nu\sigma_{+,\ \mu\rho}^{\mu\rho}$.

We observe that all the indices are summed over with only ν being a free index, so the rest are dummy indices that we have the freedom to rename. Also we have to calculate the following two:

(a) $\partial_\mu\partial_\lambda\partial_\rho\eta^{\lambda\rho}\eta^{\mu\nu}\sigma_{+a}^a = \partial^\nu\partial^\rho\partial_\rho\sigma_{+a}^a = \partial^\nu\Box\sigma_{+a}^a$.

(b) $\partial_\mu\partial_\lambda\partial_\rho\eta^{\lambda\mu}\eta^{\rho\nu}\sigma_{+a}^a = \partial^\lambda\partial^\nu\partial_\lambda\sigma_{+a}^a = \partial^\nu\Box\sigma_{+a}^a$.

So it if we plug these into $\partial_\mu\partial_\lambda\partial_\rho X^{\lambda\rho\mu\nu}$ we get:

$$\begin{aligned}\partial_\mu\partial_\lambda\partial_\rho X^{\lambda\rho\mu\nu} &= \frac{2}{d-2}\left(\Box\sigma_{+,\ \mu}^{\mu\nu} - \Box\sigma_{+,\ \mu}^{\mu\nu} - \partial^\nu\sigma_{+,\ \mu\rho}^{\mu\rho} + \partial^\nu\sigma_{+,\ \mu\rho}^{\mu\rho} - \frac{1}{d-1}\big(\partial^\nu\Box - \partial^\nu\Box\big)\sigma_{+a}^a\right)\\ &= 0.\end{aligned} \tag{16}$$

Which is the requested result.

2. *Show that the term $\frac{1}{2}\partial_\lambda\partial_\rho X^{\lambda\rho\mu\nu}$ is symmetric under $\mu \leftrightarrow \nu$.*

In order to show that it is symmetric under $\mu \leftrightarrow \nu$, it is enough to show that $\partial_\lambda\partial_\rho X^{\lambda\rho\mu\nu} - \partial_\lambda\partial_\rho X^{\lambda\rho\nu\mu} = 0$.
The starting point is the following equation, given in page (103) of Conformal Field Theory of Di Francesco:

$$X^{\lambda\rho\mu\nu} - X^{\lambda\rho\nu\mu} = \frac{2}{(d-2)(d-1)}\sigma_{+a}^a\big(\eta^{\lambda\mu}\eta^{\rho\nu} - \eta^{\lambda\nu}\eta^{\rho\mu}\big). \tag{17}$$

Then, it is not hard to show that :

$$\begin{aligned}\partial_\lambda\partial_\rho\left(X^{\lambda\rho\mu\nu} - X^{\lambda\rho\nu\mu}\right) &\simeq \partial_\lambda\partial_\rho\eta^{\lambda\mu}\eta^{\rho\nu}\sigma_{+a}^a - \partial_\lambda\partial_\rho\eta^{\lambda\nu}\eta^{\rho\mu}\sigma_{+a}^a\\ &= \partial^\mu\partial^\nu\sigma_{+a}^a - \partial^\nu\partial^\mu\sigma_{+a}^a\\ &= 0.\end{aligned} \tag{18}$$

This proves the desired result.

3. *Show that $T^{\mu\nu}$ as given in (2.26) of the notes is indeed traceless.*

We want to show that the modified energy-momentum tensor

$$T^{\mu\nu} = T_c^{\mu\nu} + \partial_\rho B^{\rho\mu\nu} + \frac{1}{2}\partial_\lambda\partial_\rho X^{\lambda\rho\mu\nu}, \tag{19}$$

is traceless, in other words that:

$$T^{\mu}_{\ \mu} = 0. \tag{20}$$

We start by multiplying eq. (19) by $\eta_{\mu\nu}$, in which case we obtain:

$$\begin{aligned} \eta_{\mu\nu}T^{\mu\nu} &= \eta_{\mu\nu}\Big(T_c^{\mu\nu} + \partial_\rho B^{\rho\mu\nu} + \frac{1}{2}\partial_\lambda\partial_\rho X^{\lambda\rho\mu\nu}\Big) \\ &= T^{\mu}_{c\ \mu} + \partial_\rho B^{\rho\mu}{}_{\mu} + \frac{1}{2}\partial_\lambda\partial_\rho X^{\lambda\rho\mu}{}_{\mu}. \end{aligned} \tag{21}$$

We need to calculate $\partial_\rho B^{\rho\mu}{}_{\mu}$ and also $\frac{1}{2}\partial_\lambda\partial_\rho X^{\lambda\rho\mu}{}_{\mu}$.

- We start with $\partial_\rho B^{\rho\mu}{}_{\mu}$. By definition, we know that :

$$B^{\rho\mu\nu} = \frac{i}{2}\left(\frac{\partial\mathcal{L}}{\partial(\partial_\rho\Phi)}S^{\nu\mu}\Phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}S^{\rho\nu}\Phi + \frac{\partial\mathcal{L}}{\partial(\partial_\nu\Phi)}S^{\rho\mu}\Phi\right). \tag{22}$$

We multiply with $\eta_{\mu\nu}$ to get:

$$\begin{aligned} B^{\rho\mu}{}_{\mu} = \eta_{\mu\nu}B^{\rho\mu\nu} &= \frac{i}{2}\left(\frac{\partial\mathcal{L}}{\partial(\partial_\rho\Phi)}S^{\mu}{}_{\mu}\Phi + \frac{\partial\mathcal{L}}{\partial(\partial^\nu\Phi)}S^{\rho\nu}\Phi + \frac{\partial\mathcal{L}}{\partial(\partial^\mu\Phi)}S^{\rho\mu}\Phi\right) \\ &= \frac{i}{2}\left(\frac{\partial\mathcal{L}}{\partial(\partial^\nu\Phi)}S^{\rho\nu}\Phi + \frac{\partial\mathcal{L}}{\partial(\partial^\mu\Phi)}S^{\rho\mu}\Phi\right) \\ &= \frac{i}{2}\left(\frac{\partial\mathcal{L}}{\partial(\partial^\mu\Phi)}S^{\rho\mu}\Phi + \frac{\partial\mathcal{L}}{\partial(\partial^\mu\Phi)}S^{\rho\mu}\Phi\right) \\ &= -i\left(\frac{\partial\mathcal{L}}{\partial(\partial^\mu\Phi)}S^{\mu\rho}\Phi\right), \end{aligned} \tag{23}$$

where in the second line we used that $S^{\mu\nu}$ is traceless, in the third line the fact that the only free index is ρ so we have the freedom to rename the dummy indices, and finally in the forth line we used that by definition $S^{\mu\nu} = -S^{\nu\mu}$. So, it is easy to compute that:

$$\partial_\rho B^{\rho\mu}{}_{\mu} = -i\partial_\rho\left(\frac{\partial\mathcal{L}}{\partial(\partial^\mu\Phi)}S^{\mu\rho}\Phi\right). \tag{24}$$

- The case of $\frac{1}{2}\partial_\lambda\partial_\rho X^{\lambda\rho\mu}{}_{\mu}$ is more lengthy. By definition we know that:

$$X^{\lambda\rho\mu\nu} = \frac{2}{d-2}\left(\eta^{\lambda\rho}\sigma_+^{\mu\nu} - \eta^{\lambda\mu}\sigma_+^{\rho\nu} - \eta^{\lambda\nu}\sigma_+^{\mu\rho} + \eta^{\mu\nu}\sigma_+^{\lambda\rho} - \frac{1}{d-1}\big(\eta^{\lambda\rho}\eta^{\mu\nu} - \eta^{\lambda\mu}\eta^{\rho\nu}\big)\sigma_{+a}^{a}\right). \tag{25}$$

So we need to compute the following:

- $\partial_\lambda \partial_\rho (\eta^{\lambda\rho}\sigma_+^{\mu\nu}) = \partial_\lambda \partial^\lambda \sigma_+^{\mu\nu} = \Box \sigma_+^{\mu\nu}$.
- $\partial_\lambda \partial_\rho (-\eta^{\lambda\mu}\sigma_+^{\rho\nu}) = -\partial^\mu \partial_\rho \sigma_+^{\rho\nu} = -\partial^\mu \partial_\lambda \sigma_+^{\lambda\nu}$.
- $\partial_\lambda \partial_\rho (-\eta^{\lambda\nu}\sigma_+^{\rho\mu}) = -\partial_\rho \partial^\nu \sigma_+^{\rho\mu} = -\partial^\nu \partial_\lambda \sigma_+^{\lambda\mu}$.
- $\partial_\lambda \partial_\rho (\eta^{\mu\nu}\sigma_+^{\lambda\rho}) = \eta^{\mu\nu} \partial_\lambda \partial_\rho \sigma_+^{\lambda\rho}$.

And we renamed some dummy indices for future convenience. Now we should contract with $\eta_{\mu\nu}$, to derive the following:

- $\eta_{\mu\nu}\partial_\lambda \partial_\rho (\eta^{\lambda\rho}\sigma_+^{\mu\nu}) = \eta_{\mu\nu}\Box \sigma_+^{\mu\nu} = \Box \sigma_{+\ \mu}^{\mu}$.
- $\eta_{\mu\nu}\Big(-\eta^{\lambda\mu}\sigma_+^{\rho\nu} (-\eta^{\lambda\nu}\sigma_+^{\rho\mu}\Big) = -\eta_{\mu\nu}\partial^\mu \partial_\lambda \sigma_+^{\lambda\nu} - \eta_{\mu\nu}\partial^\nu \partial_\lambda \sigma_+^{\lambda\mu} = -2\partial_\lambda \partial_\rho \sigma_+^{\lambda\rho}$, where we renamed the dummy indices.
- $\eta_{\mu\nu}\partial_\lambda \partial_\rho (\eta^{\mu\nu}\sigma_+^{\lambda\rho}) = \eta_{\mu\nu}\eta^{\mu\nu}\partial_\lambda \partial_\rho \sigma_+^{\lambda\rho} = d\partial_\lambda \partial_\rho \sigma_+^{\lambda\rho}$.

Hence,

$$\begin{aligned}
&\frac{2}{d-2}\eta_{\mu\nu}\partial_\lambda \partial_\rho \Big(\eta^{\lambda\rho}\sigma_+^{\mu\nu} - \eta^{\lambda\mu}\sigma_+^{\rho\nu} - \eta^{\lambda\nu}\sigma_+^{\mu\rho} + \eta^{\mu\nu}\sigma_+^{\lambda\rho}\Big) \\
&\qquad = \frac{2}{d-2}\Big(\Box \sigma_{+\ \mu}^{\mu} - 2\partial_\lambda \partial_\rho \sigma_+^{\lambda\rho} + d\partial_\lambda \partial_\rho \sigma_+^{\lambda\rho}\Big) \\
&\qquad = 2\partial_\lambda \partial_\rho \sigma_+^{\lambda\rho} + \frac{2}{d-2}\Box \sigma_{+\ \mu}^{\mu}.
\end{aligned} \tag{26}$$

Now we need to compute the following:

- $\eta_{\mu\nu}\partial_\lambda \partial_\rho (\eta^{\lambda\rho}\eta^{\mu\nu}\sigma_{+\ \mu}^{\mu}) = d\eta^{\lambda\rho}\partial_\lambda \partial_\rho \sigma_{+\ \mu}^{\mu} = d\partial_\lambda \partial^\lambda \sigma_{+\ \mu}^{\mu} = d\Box \sigma_{+\ \mu}^{\mu}$.
- $\eta_{\mu\nu}\partial_\lambda \partial_\rho (\eta^{\lambda\mu}\eta^{\rho\nu}\sigma_{+\ \mu}^{\mu}) = \eta_{\mu\nu}\eta^{\lambda\mu}\eta^{\rho\nu}\partial_\lambda \partial_\rho \sigma_{+\ \mu}^{\mu} = \delta^\lambda{}_\nu \eta^{\rho\nu}\partial_\lambda \partial_\rho \sigma_{+\ \mu}^{\mu} = \eta^{\lambda\rho}\partial_\lambda \partial_\rho \sigma_{+\ \mu}^{\mu} = \Box \sigma_{+\ \mu}^{\mu}$.

Therefore,

$$-\frac{2}{(d-1)(d-2)}\eta_{\mu\nu}\partial_\lambda \partial_\rho \Bigg(\Big(\eta^{\lambda\rho}\eta^{\mu\nu} - \eta^{\lambda\mu}\eta^{\rho\nu}\Big)\sigma_{+\ \mu}^{\mu}\Bigg) = -\frac{2}{d-2}\Box \sigma_{+\ \mu}^{\mu}, \tag{27}$$

So finally,

$$\begin{aligned}
\eta_{\mu\nu}\partial_\lambda \partial_\rho X^{\lambda\rho\mu\nu} = 2\partial_\lambda \partial_\rho \sigma_+^{\lambda\rho} &= \partial_\lambda \partial_\rho (\sigma^{\lambda\rho} + \sigma^{\rho\lambda}) \\
&= 2\partial_\lambda \partial_\rho \sigma^{\lambda\rho} = 2\partial_\rho \Big(\partial_\lambda \sigma^{\lambda\rho}\Big) \\
&= 2\partial_\rho V^\rho.
\end{aligned} \tag{28}$$

In the first line we used the definition of $\sigma_+^{\lambda\rho}$, and to go from the first line to the second, we used that fact that due to the symmetry of $\partial_\lambda \partial_\rho$, the antisymmetric part of $\sigma^{\rho\lambda}$ is zero and only the symmetric part remains, because when we multiply something symmetric with something antisymmetric the result is zero.
This gives us the desired result:

$$\eta_{\mu\nu}\frac{1}{2}\partial_\lambda \partial_\rho X^{\lambda\rho\mu\nu} = \partial_\mu V^\mu. \tag{29}$$

The V^μ is known as the virial and by definition, it is given by

$$\begin{aligned} V^\mu &= \frac{\partial \mathcal{L}}{\partial(\partial^\rho \Phi)}\left(\eta^{\mu\rho}\Delta + iS^{\mu\rho}\right)\Phi \\ &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)}\Delta\Phi + i\frac{\partial \mathcal{L}}{\partial(\partial^\rho \Phi)}S^{\mu\rho}\Phi. \end{aligned} \tag{30}$$

Then,

$$\partial_\mu V^\mu = \Delta\partial_\mu\Big(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)}\Phi\Big) + i\partial_\mu\Big(\frac{\partial \mathcal{L}}{\partial(\partial^\rho \Phi)}S^{\mu\rho}\Phi\Big). \tag{31}$$

The final requirement comes from the hypothesis that the current is conserved, i.e. $\partial_\mu j^\mu_D = 0$, which gives

$$\eta_{\mu\nu}T^{\mu\nu}_c = -\Delta\partial_\mu\Big(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)}\Phi\Big). \tag{32}$$

If we plug eq (24), eq. (31) and eq. (32) into eq. (21), we get that :

$$\begin{aligned} \eta_{\mu\nu}T^{\mu\nu} &= T^{\mu}_{c\ \mu} + \partial_\rho B^{\rho\mu}{}_\mu + \frac{1}{2}\partial_\lambda\partial_\rho X^{\lambda\rho\mu}{}_\mu \\ &= -\Delta\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)}\Phi\right) - i\partial_\rho\left(\frac{\partial \mathcal{L}}{\partial(\partial^\mu \Phi)}S^{\mu\rho}\Phi\right) \\ &+ \Delta\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)}\Phi\right) + i\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial^\rho \Phi)}S^{\mu\rho}\Phi\right) \\ &= -i\partial_\rho\left(\frac{\partial \mathcal{L}}{\partial(\partial^\mu \Phi)}S^{\mu\rho}\Phi\right) + i\partial_\rho\delta^\rho{}_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial^\rho \Phi)}S^{\mu\rho}\Phi\right) \\ &= -i\partial_\rho\left(\frac{\partial \mathcal{L}}{\partial(\partial^\mu \Phi)}S^{\mu\rho}\Phi\right) + i\partial_\rho\left(\frac{\partial \mathcal{L}}{\partial(\partial^\mu \Phi)}S^{\mu\rho}\Phi\right) \\ &= 0. \end{aligned} \tag{33}$$

And we showed that the modified energy-momentum tensor is indeed traceless, as long as we have current conservation.