Series 4

I. Correlation functions

1. *Verify the general form of the 3-pt function (2.32) (by checking first (2.29) to (2.31)).*

We will start with the two-point function to show explicitly what happens so that afterwards we can expand our techniques to the three- and four-point functions. The starting point is how an n-point function transforms:

$$\langle \phi_1(x_1)\phi_2(x_2)\dots\phi_n(x_n)\rangle = \left|\frac{\partial x'}{\partial x}\right|_{x=x_1}^{\Delta_1/d}\left|\frac{\partial x'}{\partial x}\right|_{x=x_2}^{\Delta_2/d}\cdots\left|\frac{\partial x'}{\partial x}\right|_{x=x_n}^{\Delta_n/d}\langle \phi_1(x_1')\phi_2(x_2')\dots\phi_n(x_n')\rangle. \tag{1}$$

We want to check this for the four possible symmetries of CFT : translations, i.e. $x' = x + a$, Lorentz transformations, i.e $x' = \Lambda x$, dilatations, i.e. $x' = \lambda x$ and SCTs, i.e. $x' = \frac{x-bx^2}{1-2bx+b^2x^2}$.

(a) We start with translations. From eq. (1) we derive that:

$$\begin{aligned}\langle \phi_1(x_1^\mu)\phi_2(x_2^\nu)\rangle &= \delta^\mu{}_\rho\delta^\nu{}_\sigma\langle \phi_1(x_1^\rho + a^\rho)\phi_2(x_2^\sigma + a^\sigma)\rangle\\ &= \langle \phi_1(x_1^\mu + a^\mu)\phi_2(x_2^\nu + a^\nu)\rangle.\end{aligned} \tag{2}$$

From translation invariance we deduce that if we define our two-point function as the kernel $G(x_1, x_2) \equiv \langle \phi_1(x_1)\phi_2(x_2)\rangle$, then

$$\begin{aligned}G(x_1, x_2) &= G(x_1 + a, x_2 + a)\\ &= G(x_1 + a - x_2 - a, x_2 + a - x_2 - a)\\ &= G(x_1 - x_2, 0).\end{aligned} \tag{3}$$

Thus the propagator depends only in the difference between x_1 & x_2.

(b) From the Lorentz transformations we get that:

$$\begin{aligned}\langle \phi_1(x_1^\mu)\phi_2(x_2^\nu)\rangle &= \left|\Lambda^\mu{}_\rho\right|^{\Delta_1}\left|\Lambda^\nu{}_\sigma\right|^{\Delta_2}\langle \phi_1(\Lambda^\mu{}_\rho x_1^\rho)\phi_2(\Lambda^\nu{}_\sigma x_2^\sigma)\rangle\\ &= \left(\det\Lambda^\mu{}_\rho\right)^{\Delta_1}\left(\det\Lambda^\nu{}_\sigma\right)^{\Delta_2}\langle \phi_1(\Lambda^\mu{}_\rho x_1^\rho)\phi_2(\Lambda^\nu{}_\sigma x_2^\sigma)\rangle\\ &= \langle \phi_1(\Lambda^\mu{}_\rho x_1^\rho)\phi_2(\Lambda^\nu{}_\sigma x_2^\sigma)\rangle,\end{aligned} \tag{4}$$

since $\det\Lambda = 1$.
Combining the first two transformations, we know of one quantity that depends on the difference between two points and it is also Lorentz invariant, and this is the absolute value of the interval between two different spacetime points, i.e. $|x_1 - x_2| \equiv \sqrt{\eta_{\mu\nu}\left(x_1^\mu - x_2^\mu\right)\left(x_1^\nu - x_2^\nu\right)}$.
It is obvious that this quantity depends in the difference between x_1 & x_2.
It is not hard to see that if we pick a different frame, let's say $x' = \Lambda x$

we have,

$$\begin{aligned}
|x_1' - x_2'| &= \sqrt{\eta_{\mu\nu}\left(x_1'^{\mu} - x_2'^{\mu}\right)\left(x_1'^{\nu} - x_2'^{\nu}\right)} \\
&= \sqrt{\eta_{\mu\nu}\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}\left(x_1^{\rho} - x_2^{\rho}\right)\left(x_1^{\sigma} - x_2^{\sigma}\right)} \\
&= \sqrt{\eta_{\rho\sigma}\left(x_1^{\rho} - x_2^{\rho}\right)\left(x_1^{\sigma} - x_2^{\sigma}\right)} \\
&= |x_1 - x_2|\,.
\end{aligned} \tag{5}$$

From here we can write that the most general form of the two-point function up to this point and up to trivial coefficients is:

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = C_{12}|x_1 - x_2|^{\alpha}, \tag{6}$$

An important note that has to be made here, is that the result for the first two symmetries is the same, independently of the order of the function, i.e. they result holds for two-point functions and for n-point functions.[1] This has to do with the fact that the Jacobian equals one. As we will see, this is not the case for dilatations and SCTs as these behave differently for two-point functions, three-point functions, etc, and it is these two transformations that put such firm constraints in the form of the two and three point functions.

(c) From the dilatation invariance we have that:

$$\begin{aligned}
\langle \phi_1(x_1)\phi_2(x_2))\rangle &= \lambda^{\Delta_1}\lambda^{\Delta_2}\langle \phi_1(\lambda x_1)\phi_2(\lambda x_2)\rangle \\
&= \lambda^{\Delta_1+\Delta_2}\langle \phi_1(\lambda x_1)\phi_2(\lambda x_2)\rangle.
\end{aligned} \tag{7}$$

But, we can use explicitly eq. (6) to find that for $x' = \lambda x$:

$$\begin{aligned}
\langle \phi_1(\lambda x_1)\phi_2(\lambda x_2)\rangle &= C_{12}|\lambda x_1 - \lambda x_2|^{\alpha} \\
&= C_{12}\sqrt{\eta_{\mu\nu}\left(\lambda x_1^{\mu} - \lambda x_2^{\mu}\right)\left(\lambda x_1^{\nu} - \lambda x_2^{\nu}\right)}^{\alpha} \\
&= \lambda^{\alpha} C_{12}\sqrt{\eta_{\mu\nu}\left(x_1^{\mu} - x_2^{\mu}\right)\left(x_1^{\nu} - x_2^{\nu}\right)} \\
&= \lambda^{\alpha} C_{12}|x_1 - x_2|^{\alpha} \\
&= \lambda^{\alpha}\langle \phi_1(x_1)\phi_2(x_2)\rangle.
\end{aligned} \tag{8}$$

But now, using eq. (7) and eq.(8), for the equality to stand, we see that the following condition must hold:

$$\alpha = -\left(\Delta_1 + \Delta_2\right). \tag{9}$$

And so, we can write the two-point function as:

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \frac{C_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}}. \tag{10}$$

[1] Of course for higher point functions, the general form, i.e. eq.(6) changes, as it is a product of the difference of all the points.

(d) Applying the SCT and using that $\left|\frac{\partial x'}{\partial x}\right| = \frac{1}{(1-2bx+b^2x^2)^d} = \frac{1}{{\gamma_i}^d}$, we get the following:

$$\begin{aligned}\langle \phi_1(x_1)\phi_2(x_2)\rangle &= \left(\frac{1}{(1-2bx+b^2x^2)^d}\right)^{\Delta_1/d}_{x=x_1}\left(\frac{1}{(1-2bx+b^2x^2)^d}\right)^{\Delta_2/d}_{x=x_2}\langle \phi_1(x_1')\phi_2(x_2')\rangle \\ &= \frac{1}{{\gamma_1}^{\Delta_1}}\frac{1}{{\gamma_2}^{\Delta_2}}\frac{C_{12}}{\left|x_1'-x_2'\right|^{\Delta_1+\Delta_2}}.\end{aligned} \tag{11}$$

At this point, we should calculate how the absolute value of the interval between the two points changes under SCT.

$$\begin{aligned}\left|x_1'-x_2'\right| &= \sqrt{\eta_{\mu\nu}\left(x_1'^{\mu}-x_2'^{\mu}\right)\left(x_1'^{\nu}-x_2'^{\nu}\right)} \\ &= \sqrt{x_1'^{\mu}x'_{1\,\mu} - x_1'^{\mu}x'_{2\,\mu} - x_2'^{\mu}x'_{1\,\mu} + x_2'^{\mu}x'_{2\,\mu}} \\ &= \sqrt{\frac{\left(x_1^{\mu}-b^{\mu}{x_1}^2\right)}{\gamma_1}\frac{\left(x_{1\mu}-b_{\mu}{x_1}^2\right)}{\gamma_1} + \frac{\left(x_2^{\mu}-b^{\mu}{x_2}^2\right)}{\gamma_2}\frac{\left(x_{2\mu}-b_{\mu}{x_2}^2\right)}{\gamma_2} - \frac{\omega}{\gamma_1\gamma_2}} \\ &= \sqrt{\frac{{x_1}^2(1-2bx_1+b^2{x_1}^2)}{\gamma_1(1-2bx_1+b^2{x_1}^2)} + \frac{{x_2}^2(1-2bx_2+b^2{x_2}^2)}{\gamma_2(1-2bx_2+b^2{x_2}^2)} - \frac{\omega}{\gamma_1\gamma_2}} \\ &= \sqrt{\frac{{x_1}^2}{\gamma_1}+\frac{{x_2}^2}{\gamma_2}-\frac{\omega}{\gamma_1\gamma_2}} \\ &= \sqrt{\frac{{x_1}^2\gamma_2+{x_2}^2\gamma_1-\omega}{\gamma_1\gamma_2}} \\ &= \sqrt{\frac{{x_1}^2+{x_2}^2-2x_1x_2}{\gamma_1\gamma_2}} \\ &= \frac{|x_1-x_2|}{\sqrt{\gamma_1\gamma_2}},\end{aligned} \tag{12}$$

where we set that $\omega = 2\left(x_1^{\mu}-b^{\mu}{x_1}^2\right)\left(x_{2\mu}-b_{\mu}{x_2}^2\right)$. Hence, we can find that

$$\frac{1}{\left|x_1'-x_2'\right|^{\Delta_1+\Delta_2}} = \frac{1}{|x_1-x_2|^{\Delta_1+\Delta_2}}{\gamma_1}^{\frac{\Delta_1+\Delta_2}{2}}{\gamma_2}^{\frac{\Delta_1+\Delta_2}{2}} \tag{13}$$

Hence, plugging eq. (13) into eq. (11) we find that:

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \frac{C_{12}}{|x_1-x_2|^{\Delta_1+\Delta_2}}{\gamma_1}^{\frac{\Delta_1+\Delta_2}{2}}{\gamma_2}^{\frac{\Delta_1+\Delta_2}{2}}\frac{1}{{\gamma_1}^{\Delta_1}}\frac{1}{{\gamma_2}^{\Delta_2}}. \tag{14}$$

Thus, we have the following matching conditions:

- $\frac{\Delta_1+\Delta_2}{2} - \Delta_1 = 0$,
- $\frac{\Delta_1+\Delta_2}{2} - \Delta_2 = 0$,

which have the unique solution that

$$\Delta_1 = \Delta_2 = \Delta. \tag{15}$$

Finally, we can normalize the fields in such a way that $C_{12} = \delta_{12}$ and the final form of the two-point function is

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \frac{1}{|x_1 - x_2|^{2\Delta}}, \tag{16}$$

where the only unknown is the scaling dimension Δ of the fields. We can see that this is very strong constraint, which is the immense power of CFT.

The reason that we underwent so much trouble, is that by doing the full analysis in the simplest case, it is much easier to proceed for the three- and afterwards the four-point functions.
So, in the same spirit as before, we should start with eq. (1), but with three fields now, and repeat all the steps. But, we have seen that the translation and Lorentz invariance apply the same to all n point functions, and thus we can deduce that the three point function is determined by the absolute value of the interval between the three points, in some power, in other words:

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = C_{123}|x_1 - x_2|^{\alpha}|x_2 - x_3|^{b}|x_1 - x_3|^{c}, \tag{17}$$

where from now on we will use $|x_{12}| = |x_1 - x_2|$ for shortness. Let's see now what happens under dilatations and SCTs.

(a) From dilatation we have:

$$\begin{aligned}\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle &= \lambda^{\Delta_1+\Delta_2+\Delta_3}\langle \phi_1(\lambda x_1)\phi_2(\lambda x_2)\phi_3(\lambda x_3)\rangle \\ &= \lambda^{\Delta_1+\Delta_2+\Delta_3}C_{123}|\lambda x_{12}|^{\alpha}|\lambda x_{23}|^{b}|\lambda x_{13}|^{c} \\ &= \lambda^{\Delta_1+\Delta_2+\Delta_3+\alpha+b+c}C_{123}|x_{12}|^{\alpha}|x_{23}|^{b}|x_{13}|^{c} \\ &= \lambda^{\Delta_1+\Delta_2+\Delta_3+\alpha+b+c}\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle.\end{aligned} \tag{18}$$

Thus, we obtain the constraint that:

$$\Delta_1 + \Delta_2 + \Delta_3 + \alpha + b + c = 0 \Leftrightarrow \Delta_1 + \Delta_2 + \Delta_3 = -\alpha - b - c. \tag{19}$$

(b) Under SCT, and following eq.(11), we have that

$$\begin{aligned}\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle &= \frac{1}{{\gamma_1}^{\Delta_1}}\frac{1}{{\gamma_2}^{\Delta_2}}\frac{1}{{\gamma_3}^{\Delta_3}}C_{123}\left|x'_{12}\right|^{\alpha}\left|x'_{23}\right|^{b}\left|x'_{13}\right|^{c} \\ &= \frac{C_{123}}{{\gamma_1}^{\Delta_1}{\gamma_2}^{\Delta_2}{\gamma_3}^{\Delta_3}}\frac{|x_{12}|^{\alpha}}{(\gamma_1\gamma_2)^{\alpha/2}}\frac{|x_{23}|^{b}}{(\gamma_2\gamma_3)^{b/2}}\frac{|x_{13}|^{c}}{(\gamma_1\gamma_3)^{c/2}}.\end{aligned} \tag{20}$$

and to go from the second to the third line, we used eq.(12). From this we find the following three constraints:

$$\Delta_1 + \alpha/2 + c/2 = 0, \tag{21}$$
$$\Delta_2 + \alpha/2 + b/2 = 0, \tag{22}$$
$$\Delta_3 + b/2 + c/2 = 0. \tag{23}$$

We can solve for α, b, c to get:

$$\alpha = \Delta_3 - \Delta_1 - \Delta_2, \tag{24}$$
$$b = \Delta_1 - \Delta_2 - \Delta_3, \tag{25}$$
$$c = \Delta_2 - \Delta_1 - \Delta_3. \tag{26}$$

We can check explicitly that this unique set of solutions satisfies eq. (19). Using these, the final form of the three-point function is:

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \frac{C_{123}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}\,|x_{23}|^{\Delta_2+\Delta_3-\Delta_1}\,|x_{13}|^{\Delta_1+\Delta_3-\Delta_2}}. \tag{27}$$

Since we have normalized the fields for the two-point function, we cannot normalize them again, hence C_{123} is an important part of the CFT. So we can see that if we know the scaling dimension of the fields (which can be calculated through the two-point function), the only unknown part of the three-point function are the OPE coefficients. So once more we observe that the conformal transformation impose some very strong constraints on the form of the three-point function.

2. *Verify the general form of the 4-pt function (2.33) in a similar fashion.*

For the four-point function, again we can use the same logic as for the three-point function, and by using translation and Lorentz invariance, we can write it in the form:

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle = C_{1234}\,|x_{12}|^{\alpha}\,|x_{13}|^{b}\,|x_{14}|^{c}\,|x_{23}|^{d}\,|x_{24}|^{\epsilon}\,|x_{34}|^{z}. \tag{28}$$

As we will soon see, the coefficients in front are not so innocent as the ones of the two and three-point functions. For the time being, we can concentrate to the dilatation invariance, which is exactly the same as the three point function but with more coefficients. Thus,

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle = \lambda^{\Delta_1+\Delta_2+\Delta_3+\Delta_4+\alpha+b+c+d+\epsilon+z}\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle, \tag{29}$$

which gives the constraint

$$\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 = -\big(\alpha + b + c + d + \epsilon + z\big). \tag{30}$$

From the SCT invariance, and using eq.(12) we have:

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle = \frac{C_{1234}}{\gamma_1{}^{\Delta_1}\gamma_2{}^{\Delta_2}\gamma_3{}^{\Delta_3}\gamma_4{}^{\Delta_4}}\frac{|x_{12}|^{\alpha}}{(\gamma_1\gamma_2)^{\alpha/2}}\frac{|x_{13}|^{b}}{(\gamma_1\gamma_3)^{b/2}}\frac{|x_{14}|^{c}}{(\gamma_1\gamma_4)^{c/2}}\frac{|x_{23}|^{d}}{(\gamma_2\gamma_3)^{d/2}}\frac{|x_{24}|^{d}}{(\gamma_2\gamma_4)^{\epsilon/2}}\frac{|x_{34}|^{d}}{(\gamma_3\gamma_4)^{z/2}}. \tag{31}$$

By matching the gamma, we derive the following conditions:

$$\alpha + b + c = -2\Delta_1, \tag{32}$$
$$\alpha + d + \epsilon = -2\Delta_2, \tag{33}$$
$$b + d + z = -2\Delta_3, \tag{34}$$
$$c + \epsilon + z = -2\Delta_4. \tag{35}$$

But this set of equations is impossible to be solved as we have six unknowns for four equations. But for the four-point function there is a catch. When having four points and more, it is possible to create certain coefficients that preserve the CFT symmetries. The trick is to use the absolute value of the interval of two points, which is a priori Lorentz and translation invariant. Thus, the catch is to find the correct combination that will preserve dilatation and SCT invariance. It is clear that this is not the case for two and three point functions. The two-point function is too simple as it contains only one interval. For the three-point function it is impossible to create any coefficient that is invariant under dilatation, e.g. for $\chi = \frac{|x_{12}||x_{23}|}{|x_{13}|}$ scales like λ under dilatations, $\zeta = \frac{|x_{12}|}{|x_{13}|}$ is invariant under dilatations but it is not invariant under SCT, etc. But for the four point function we can create some cross ratios, which are

$$\chi_1 = \frac{|x_{12}|\,|x_{34}|}{|x_{13}|\,|x_{24}|}, \quad \chi_2 = \frac{|x_{12}|\,|x_{34}|}{|x_{23}|\,|x_{14}|}. \tag{36}$$

The dilatation invariance is obvious. What is interesting is that in order for χ_1 & χ_2 to be SCT invariant, we gain two more equations or more precisely, constraints (remember from our analysis above, we had more unknowns than equations, which meant that we had some freedom in the scaling). Thus from χ_1 & χ_2 if we SCT transform them as usual, we derive that

$$b + \epsilon = \alpha + z, \tag{37}$$
$$d + c = \alpha + z. \tag{38}$$

These two, combined with equations (32 - 35) lead to:

$$2\alpha + z - \epsilon + c = -2\Delta_1 \tag{39}$$
$$2\alpha + z - c + \epsilon = -2\Delta_2 \tag{40}$$
$$2\alpha + 3z - c - \epsilon = -2\Delta_3 \tag{41}$$
$$c + \epsilon + z = -2\Delta_4 \tag{42}$$

Now it is obvious that we have the correct number of unknowns and equations, thus by solving the system we get the following results:

$$\alpha = -\frac{2\Delta_1}{3} - \frac{2\Delta_2}{3} + \frac{\Delta_3}{3} + \frac{\Delta_4}{3} \tag{43}$$
$$b = -\frac{2\Delta_1}{3} - \frac{2\Delta_3}{3} + \frac{\Delta_4}{3} + \frac{\Delta_2}{3} \tag{44}$$
$$c = -\frac{2\Delta_1}{3} - \frac{2\Delta_4}{3} + \frac{\Delta_2}{3} + \frac{\Delta_3}{3} \tag{45}$$
$$d = -\frac{2\Delta_2}{3} - \frac{2\Delta_3}{3} + \frac{\Delta_4}{3} + \frac{\Delta_1}{3} \tag{46}$$
$$\epsilon = -\frac{2\Delta_2}{3} - \frac{2\Delta_4}{3} + \frac{\Delta_1}{3} + \frac{\Delta_3}{3} \tag{47}$$
$$z = -\frac{2\Delta_3}{3} - \frac{2\Delta_4}{3} + \frac{\Delta_1}{3} + \frac{\Delta_2}{3}. \tag{48}$$

In short notation, these can be written as

$$\Delta/3 - \Delta_i - \Delta_j, \quad \Delta = \sum_{i=1}^{4} \Delta_i. \tag{49}$$

Hence, the four-point function can be written in terms of conformal blocks:

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle = \mathcal{F}(\chi_1, \chi_2) \prod_{i<j}^{4} x_{ij}^{\Delta/3-\Delta_i-\Delta_j}, \tag{50}$$

where $\mathcal{F}(\chi_1, \chi_2)$ is a function of all possible cross ratios.

II. Ward identities

1. *Derive the reduced form of the Ward identity for dilation (2.37) from (2.36) using the Ward identity for translation (2.34). Note: you can pull x out of the path integral!*

Our starting point is that

$$\partial_\mu \langle T^\mu{}_\nu x^\nu X\rangle = -\sum_i \delta(x - x_i)\left\{ x_i^\nu \frac{\partial}{\partial x_i^\nu}\langle X\rangle + \Delta_i \langle X\rangle \right\}. \tag{51}$$

We focus on the left-hand side of eq. (51), thus:

$$\begin{aligned} \partial_\mu \langle T^\mu{}_\nu x^\nu X\rangle &= \langle (\partial_\mu T^\mu{}_\nu) x^\nu X + \partial_\mu x^\nu T^\mu_\nu X\rangle \\ &= x^\nu \partial_\mu \langle T^\mu{}_\nu X\rangle + \langle \delta_\mu{}^\nu T^\mu{}_\nu X\rangle \\ &= x^\nu \partial_\mu \langle T^\mu{}_\nu X\rangle + \langle T^\mu{}_\mu X\rangle. \end{aligned} \tag{52}$$

But then we can use that the Ward identity associated with translation is

$$\partial_\mu \langle T^\mu{}_\nu X\rangle = -\sum_i \delta(x - x_i)\frac{\partial}{\partial x_i^\nu}\langle X\rangle. \tag{53}$$

Combining eq. (51), eq. (52) and eq. (53) we get that :

$$\partial_\mu \langle T^\mu{}_\nu x^\nu X\rangle = -\sum_i \delta(x-x_i)\left\{x_i^\nu \frac{\partial}{\partial x_i^\nu}\langle X\rangle + \Delta_i\langle X\rangle\right\} \Leftrightarrow \tag{54}$$

$$x^\nu \partial_\mu \langle T^\mu{}_\nu X\rangle + \langle T^\mu{}_\mu X\rangle = -\sum_i \delta(x-x_i)\left\{x_i^\nu \frac{\partial}{\partial x_i^\nu}\langle X\rangle + \Delta_i\langle X\rangle\right\} \Leftrightarrow \tag{55}$$

$$-\sum_i \delta(x-x_i)\frac{\partial}{\partial x_i^\nu}\langle X\rangle + \langle T^\mu{}_\mu X\rangle = -\sum_i \delta(x-x_i)\left\{x_i^\nu \frac{\partial}{\partial x_i^\nu}\langle X\rangle + \Delta_i\langle X\rangle\right\} \Leftrightarrow \tag{56}$$

$$\langle T^\mu{}_\mu X\rangle = -\sum_i \delta(x-x_i)\Delta_i\langle X\rangle, \tag{57}$$

which is the desired result.