Series 5

I. 3-point fun of scalar quasi-primaries using null cone formalism

1. *Rederive the form of the 3-pt function, this time using the null-cone formalism, following the arguments for the 2-pt function in the notes.*

The most general form of the three-point function on the light cone, where we know that $X_1^2 = X_2^2 = X_3^2 = 0$ is:

$$\langle \phi_1(X_1)\phi_2(X_2)\phi_3(X_3)\rangle = \frac{Const.}{(X_1X_2)^{\alpha}(X_1X_3)^{b}(X_2X_3)^{c}}. \tag{1}$$

The system has to be consistent with scaling, for this reason we should impose the following scaling constraints, which are valid under a dilatation:

$$\alpha + b = \Delta_1, \tag{2}$$
$$\alpha + c = \Delta_2, \tag{3}$$
$$b + c = \Delta_3. \tag{4}$$

This system can be solved explicitly and admits the following unique solutions for α, b, c:

$$\alpha = \frac{\Delta_1 + \Delta_2 - \Delta_3}{2}, \tag{5}$$
$$b = \frac{\Delta_1 + \Delta_3 - \Delta_2}{2}, \tag{6}$$
$$c = \frac{\Delta_2 + \Delta_3 - \Delta_1}{2}. \tag{7}$$

In order to find the form of the three-point function in $\mathcal{R}^D$, we have to project X_1, X_2, X_3 to the following sections:

$$X_1{}^M = (X_1{}^+, X_1{}^-, X_1{}^\mu) = (1, x_1{}^2, x_1{}^\mu), \tag{8}$$
$$X_2{}^M = (X_2{}^+, X_2{}^-, X_2{}^\mu) = (1, x_2{}^2, x_2{}^\mu), \tag{9}$$
$$X_3{}^M = (X_3{}^+, X_3{}^-, X_3{}^\mu) = (1, x_3{}^2, x_3{}^\mu). \tag{10}$$

Then,

$$\begin{aligned} X_1X_2 &= \delta_{\mu\nu}X_1{}^\mu X_2{}^\mu + \eta_{+-}X_1{}^+X_2{}^- + \eta_{-+}X_1{}^-X_2{}^+ \\ &= X_1{}^\mu X_{2\mu} - \frac{1}{2}X_1{}^+X_2{}^- - \frac{1}{2}X_1{}^-X_2{}^+ \\ &= x_1{}^\mu x_{2\mu} - \frac{1}{2}(x_2{}^2 + x_1{}^2) \\ &= -\frac{1}{2}(x_1 - x_2)^2 \\ &= -\frac{1}{2}(x_{12})^2, \end{aligned} \tag{11}$$

where $x_{ij} = x_1 - x_j$. Similarly,

$$X_1 X_3 = -\frac{1}{2}(x_{13})^2, \tag{12}$$

$$X_2 X_3 = -\frac{1}{2}(x_{23})^2. \tag{13}$$

Also since $\phi(x) = \phi(X)$, we get:

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \frac{C_{123}}{(x_{12})^{2\alpha}(x_{13})^{2b}(x_{23})^{2c}}. \tag{14}$$

Thus, plugging in α, b, c from above we get back the familiar expression

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \frac{C_{123}}{(x_{12})^{2\frac{\Delta_1+\Delta_2-\Delta_3}{2}}(x_{13})^{2\frac{\Delta_1+\Delta_3-\Delta_2}{2}}(x_{23})^{2\frac{\Delta_2+\Delta_3-\Delta_1}{2}}}. \tag{15}$$

II. Higher spin fields in the null-cone formalism

1. *Show that tracelessness is preserved by the projection on the section (2.45). To do so, show that*

$$\delta^{\mu\nu}\frac{\partial X^M}{\partial x^\mu}\frac{\partial X^N}{\partial x^\nu} = \eta^{MN} + X^M K^N + X^N K^M, \tag{16}$$

with $K^M = (0, 2, 0)$. *Complete the argument using this result.*

We will start by showing that $\delta^{\mu\nu}\frac{\partial X^M}{\partial x^\mu}\frac{\partial X^N}{\partial x^\nu} = \eta^{MN} + X^M K^N + X^N K^M$, where $X^M = (1, x^2, x^\mu)$, $K^M = (0, 2, 0)$ and $\frac{\partial X^M}{\partial x^\mu} = (0, 2x_\mu, \delta^\mu{}_\mu)$.
This proof is not constructive, in other words, we do not start in the l.h.s to go to the r.h.s but we merely show that this result stands. At the end of the exercise there will be a constructive proof, which is more mathematically rigorous.
First we observe in the r.h.s that η^{MN} is a matrix, which means that $X^M K^N$ is also a matrix and the same stands for $\delta^{\mu\nu}\frac{\partial X^M}{\partial x^\mu}\frac{\partial X^N}{\partial x^\nu}$. So we have to use the tensor product. Starting in the l.h.s we have:

$$\begin{aligned}
\delta^{\mu\nu}\frac{\partial X^M}{\partial x^\mu}\frac{\partial X^N}{\partial x^\nu} &= \delta^{\mu\nu}\frac{\partial X^M}{\partial x^\mu} \otimes \frac{\partial X^N}{\partial x^\nu} \\
&= \delta^{\mu\nu}(0, 2x_\mu, \delta^\mu{}_\mu) \otimes (0, 2x_\nu, \delta^\nu{}_\nu) \\
&= \delta^{\mu\nu}\left(4x_\mu x_\nu + 2x_\mu + 2x_\nu + \delta^\mu{}_\mu \delta^\nu{}_\nu\right) \\
&= 4x^2 + 2x^\nu + 2x^\mu + \delta^{\mu\nu}.
\end{aligned} \tag{17}$$

Now, in the r.h.s we have to check two things. First, how we can write η^{MN} and then calculate $X^M K^N$. So starting with η^{MN}, we know that

$$\eta^{MN} = \delta^{\mu\nu} + \eta^{+-} + \eta^{-+}. \tag{18}$$

But also we know that

$$\eta^{MN}\eta_{MN} = D+2, \tag{19}$$
$$\delta^{\mu\nu}\delta_{\mu\nu} = D. \tag{20}$$

Thus, combining the above results we see that:

$$\begin{aligned}
&\eta^{MN}\eta_{MN} = D+2,\\
&\delta^{\mu\nu}\delta_{\mu\nu} + \eta^{+-}\eta_{+-} + \eta^{-+}\eta_{-+} = D+2,\\
&D + \eta^{+-}\eta_{+-} + \eta^{-+}\eta_{-+} = D+2,\\
&\eta^{+-}\eta_{+-} + \eta^{-+}\eta_{-+} = 2,\\
&\eta^{+-} = -2,
\end{aligned} \tag{21}$$

where we used that $\eta_{+-} = \eta_{-+} = -\frac{1}{2}$ and that $\eta^{+-} = \eta^{-+}$. Hence,

$$\eta^{MN} = \delta^{\mu\nu} - 2 - 2. \tag{22}$$

Also,

$$\begin{aligned}
X^M K^N &= X^M \otimes K^N\\
&= (1, x^2, x^\mu) \otimes (0, 2, 0)\\
&= 2 + 2x^2 + 2x^\mu.
\end{aligned} \tag{23}$$

Similarly, $X^N K^M = 2 + 2x^2 + 2x^\nu$. Combining all these we can see that the r.h.s reads:

$$\begin{aligned}
\eta^{MN} + X^M K^N + X^N K^M &= \delta^{\mu\nu} - 2 - 2 + 2 + 2x^2 + 2x^\mu + 2 + 2x^2 + 2x^\nu\\
&= \delta^{\mu\nu} + 4x^2 + 2x^\mu + 2x^\nu\\
&= \delta^{\mu\nu}\frac{\partial X^M}{\partial x^\mu}\frac{\partial X^N}{\partial x^\nu}.
\end{aligned} \tag{24}$$

Now it is easy to show that the tracelessness is preserved,
e.g. $\delta^{\mu\nu}\phi_{\mu\nu\lambda}\ldots(x) = 0$.
By definition,

$$\phi_{\mu\nu\lambda\ldots}(x) = \phi_{MN\Lambda\ldots}(X)\frac{\partial X^M}{\partial x^\mu}\frac{\partial X^N}{\partial x^\nu}. \tag{25}$$

So, if we multiply by $\delta^{\mu\nu}$ in the l.h.s we get:

$$\begin{aligned}
\delta^{\mu\nu}\phi_{\mu\nu\lambda\ldots}(x) &= \delta^{\mu\nu}\phi_{MN\Lambda\ldots}(X)\frac{\partial X^M}{\partial x^\mu}\frac{\partial X^N}{\partial x^\nu}\\
&= \phi_{MN\Lambda\ldots}(X)\delta^{\mu\nu}\frac{\partial X^M}{\partial x^\mu}\frac{\partial X^N}{\partial x^\nu}\\
&= \phi_{MN\Lambda\ldots}(X)\left(\eta^{MN} + X^M K^N + X^N K^M\right)\\
&= \phi^N{}_{N\ldots}(X) + X^M\phi_{MN\ldots}(X)\otimes K^N + X^N\phi_{MN\ldots}(X)\otimes K^M = 0,
\end{aligned} \tag{26}$$

where ${\phi^N}_{N\Lambda\dots}(X)$ is traceless by definition and that $X^M\phi_{MN\dots}(X) = 0$ by transversality.

Rigorous proof of : $\delta^{\mu\nu}\frac{\partial X^M}{\partial x^\mu}\frac{\partial X^N}{\partial x^\nu} = \eta^{MN} + X^M K^N + X^N K^M$.

Since we are dealing with matrices, we are going to use a standard linear algebra method, where we denote the components of a vector through a base vector. In other words we should write:

$$X^M = (1\hat{\mathbf{i}}, x^2\hat{\mathbf{j}}, x^\mu\hat{\mathbf{k}}), \tag{27}$$

$$K^M = (0.2\hat{\mathbf{j}}, 0), \tag{28}$$

$$\frac{\partial X^M}{\partial x^\mu} = (0, 2x_\mu\hat{\mathbf{j}}, {\delta^\mu}_\mu\hat{\mathbf{k}}). \tag{29}$$

Thus,

$$\begin{aligned}\frac{\partial X^M}{\partial x^\mu}\frac{\partial X^N}{\partial x^\nu} &= \frac{\partial X^M}{\partial x^\mu} \otimes \frac{\partial X^N}{\partial x^\nu} \\ &= 4x_\mu x_\nu \hat{\mathbf{j}} \otimes \hat{\mathbf{j}} + 2x_\mu {\delta^\nu}_\nu \hat{\mathbf{j}} \otimes \hat{\mathbf{k}} + 2x_\nu {\delta^\mu}_\mu \hat{\mathbf{k}} \otimes \hat{\mathbf{j}} + {\delta^\mu}_\mu {\delta^\nu}_\nu \hat{\mathbf{k}} \otimes \hat{\mathbf{k}}.\end{aligned} \tag{30}$$

Hence,

$$\begin{aligned}\delta^{\mu\nu}\frac{\partial X^M}{\partial x^\mu}\frac{\partial X^N}{\partial x^\nu} &= 4x^2\hat{\mathbf{j}} \otimes \hat{\mathbf{j}} + 2x^\nu\hat{\mathbf{j}} \otimes \hat{\mathbf{k}} + 2x^\mu\hat{\mathbf{k}} \otimes \hat{\mathbf{j}} + \delta^{\mu\nu}\hat{\mathbf{k}} \otimes \hat{\mathbf{k}} \\ &= 2(x^2\hat{\mathbf{j}} + x^\mu\hat{\mathbf{k}}) \otimes \hat{\mathbf{j}} + \hat{\mathbf{j}} \otimes 2(x^2\hat{\mathbf{j}} + x^\nu\hat{\mathbf{k}}) + \delta^{\mu\nu}\hat{\mathbf{k}} \otimes \hat{\mathbf{k}} \\ &= 2(X^M - 1\hat{\mathbf{i}}) \otimes \hat{\mathbf{j}} + \hat{\mathbf{j}} \otimes 2(X^N - 1\hat{\mathbf{i}}) + \delta^{\mu\nu}\hat{\mathbf{k}} \otimes \hat{\mathbf{k}} \\ &= 2X^M \otimes \hat{\mathbf{j}} - 2\hat{\mathbf{i}} \otimes \hat{\mathbf{j}} + \hat{\mathbf{j}} \otimes 2X^N - 2\hat{\mathbf{j}} \otimes \hat{\mathbf{i}} + \delta^{\mu\nu}\hat{\mathbf{k}} \otimes \hat{\mathbf{k}} \\ &= X^M K^N + K^M X^N + \delta^{\mu\nu}\hat{\mathbf{k}} \otimes \hat{\mathbf{k}} - 2\hat{\mathbf{i}} \otimes \hat{\mathbf{j}} - 2\hat{\mathbf{j}} \otimes \hat{\mathbf{i}}.\end{aligned} \tag{31}$$

But now we should remember our definition of $X^\pm$. We have that:

$$X^+ = \frac{1}{2}\big(X^{D+2} + X^{D+1}\big), \tag{32}$$

$$X^- = \frac{1}{2}\big(X^{D+2} - X^{D+1}\big). \tag{33}$$

In other words, we have that:

$$\hat{\mathbf{i}} = \frac{1}{2}\big((\mathbf{D} \hat{+} \mathbf{2}) + (\mathbf{D} \hat{+} \mathbf{1})\big), \tag{34}$$

$$\hat{\mathbf{j}} = \frac{1}{2}\big((\mathbf{D} \hat{+} \mathbf{2}) - (\mathbf{D} \hat{+} \mathbf{1})\big). \tag{35}$$

If we use these, we can calculate $\hat{\mathbf{i}} \otimes \hat{\mathbf{j}}$ and $\hat{\mathbf{j}} \otimes \hat{\mathbf{i}}$. We find that:

$$\begin{aligned}\hat{\mathbf{i}} \otimes \hat{\mathbf{j}} = \frac{1}{4}\big((\mathbf{D} \hat{+} \mathbf{2}) \otimes (\mathbf{D} \hat{+} \mathbf{2}) &- (\mathbf{D} \hat{+} \mathbf{2}) \otimes (\mathbf{D} \hat{+} \mathbf{1}) \\ &- (\mathbf{D} \hat{+} \mathbf{1}) \otimes (\mathbf{D} \hat{+} \mathbf{1}) + (\mathbf{D} \hat{+} \mathbf{1}) \otimes (\mathbf{D} \hat{+} \mathbf{2})\big),\end{aligned} \tag{36}$$

and,

$$\hat{\mathbf{j}} \otimes \hat{\mathbf{i}} = \frac{1}{4}\big((\mathbf{D} \hat{+} \mathbf{2}) \otimes (\mathbf{D} \hat{+} \mathbf{2}) + (\mathbf{D} \hat{+} \mathbf{2}) \otimes (\mathbf{D} \hat{+} \mathbf{1}) - (\mathbf{D} \hat{+} \mathbf{1}) \otimes (\mathbf{D} \hat{+} \mathbf{1}) - (\mathbf{D} \hat{+} \mathbf{1}) \otimes (\mathbf{D} \hat{+} \mathbf{2})\big). \quad (37)$$

Thus, we can combine the above to get

$$-2\big(\hat{\mathbf{i}} \otimes \hat{\mathbf{j}} + \hat{\mathbf{j}} \otimes \hat{\mathbf{i}}\big) = (\mathbf{D} \hat{+} \mathbf{1}) \otimes (\mathbf{D} \hat{+} \mathbf{1}) - (\mathbf{D} \hat{+} \mathbf{2}) \otimes (\mathbf{D} \hat{+} \mathbf{2}). \quad (38)$$

But we observe that

$$\eta^{MN} = \delta^{\mu\nu}\hat{\mathbf{k}} \otimes \hat{\mathbf{k}} + (\mathbf{D} \hat{+} \mathbf{1}) \otimes (\mathbf{D} \hat{+} \mathbf{1}) - (\mathbf{D} \hat{+} \mathbf{2}) \otimes (\mathbf{D} \hat{+} \mathbf{2}). \quad (39)$$

In other words,

$$\eta^{MN} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \delta^{\mu\nu} \end{bmatrix}.$$

So finally we get that

$$\delta^{\mu\nu}\frac{\partial X^M}{\partial x^\mu}\frac{\partial X^N}{\partial x^\nu} = X^M K^N + X^N K^M + \eta^{MN}. \quad (40)$$

2. *Show that (2.47) and (2.48) lead to (2.49) in the notes for the case of spin 1 (vector field). To do so, show that*

$$\phi'(x') \cdot \mathrm{d}x' = \Lambda^{-(\Delta-1)/2}\phi(x) \cdot \mathrm{d}x. \quad (41)$$

Use furthermore the projection rule and that under Lorentz transformations, the scalar product $\phi(X) \cdot \mathrm{d}X$ is preserved.

We want to prove that

$$\phi'_{\mu\ldots}(x') = \Lambda(x)^{-\Delta/2} M_\mu^{\mu'}(x) \ldots \phi_{\mu'\ldots}(x), \quad (42)$$

where $M_\mu^{\mu'}(x)$ is a rotation matrix.
We know the following identities which will be used to prove the result:

$$X^M = \Lambda^M_N X^N, \quad (43)$$

$$X' = \Lambda^{1/2} Y \to Y = \Lambda^{-1/2} X', \quad (44)$$

$$\phi_\mu(x) = \phi_M(X), \quad (45)$$

$$X^M \phi_{MNL\ldots}(X) = 0, \quad (46)$$

$$\phi'_M(X') = \Lambda^{M'}{}_M \phi_{M'}(X'), \quad (47)$$

$$\phi_{MNL\ldots}(\lambda X) = \lambda^{-\Delta}\phi_{MNL\ldots}(X) \quad (48)$$

$$\mathrm{d}x' = \Lambda(x)^{1/2} M(x)\mathrm{d}x. \quad (49)$$

We start with the projection rule, thus

$$\phi_\mu(x)\mathrm{d}x^\mu = \phi_M(X)\frac{\partial X^M}{\partial x^\mu}\mathrm{d}x^\mu = \phi_M(X)\mathrm{d}X^M. \quad (50)$$

Under a Lorentz transformation, $X \to \Lambda X$, the scalar product $\phi(X) \cdot \mathrm{d}X$ is preserved, e.g. for $Y = \Lambda X$

$$\begin{aligned} \phi_M(X)\mathrm{d}X^M \mapsto \phi'_M(Y)\mathrm{d}Y^M &= \Lambda_M{}^P \phi_P(X) \Lambda^M{}_L \mathrm{d}X^L \\ &= \delta^P{}_L \phi_P(X)\mathrm{d}X^L \\ &= \phi_L(X)\mathrm{d}X^L. \end{aligned} \tag{51}$$

To get from Y back to the X section, we have to rescale, e.g. $Y = \Lambda^{-1/2}X'$. Then by using eq.(48) we get:

$$\begin{aligned} \phi_M(\Lambda^{-1/2}X') &= \left(\Lambda^{-1/2}\right)^{-\Delta} \phi_M(X') \\ &= \Lambda^{\Delta/2}\phi_M(X'). \end{aligned} \tag{52}$$

Also,

$$dY = \Lambda^{-1/2}dX' + X'd((\Lambda^{-1/2}). \tag{53}$$

Thus,

$$\begin{aligned} \phi'_M(Y)dY^M &= \Lambda^{\Delta/2}\phi'{}_M(X')\left(\Lambda^{-1/2}dX'^M + X'^M d((\Lambda^{-1/2})\right) \\ &= \Lambda^{\frac{\Delta-1}{2}}\phi'{}_M(X')dX'^M \\ &= \Lambda^{\frac{\Delta-1}{2}}\phi'{}_\mu(x')dx'^\mu, \end{aligned} \tag{54}$$

where to get from the first line to the second we used the property of transversality, e.g. eq.(46), and from the second line to the third we used the projection rule.

Now we know that the r.h.s of eq.(50) is invariant under Lorentz transformations. So the l.h.s should also be invariant under Lorentz, so combining eq. (50) and eq.(54) we get:

$$\phi_\mu(x)\mathrm{d}x^\mu = \phi'_M(Y)\mathrm{d}Y^M = \Lambda^{\frac{\Delta-1}{2}}\phi'{}_\mu(x')\mathrm{d}x'^\mu, \tag{55}$$

and thus

$$\phi'{}_\mu(x')\mathrm{d}x'^\mu = \Lambda^{-\frac{\Delta-1}{2}}\phi_\mu(x)\mathrm{d}x^\mu. \tag{56}$$

But from eq.(49) we know that

$$\mathrm{d}x'^\mu = \Lambda(x)^{1/2}M^\mu{}_{\mu'}\mathrm{d}x^{\mu'}, \tag{57}$$

and so

$$\phi'{}_\mu(x') = \Lambda^{-\frac{\Delta}{2}}M_\mu{}^{\mu'}\phi_{\mu'}(x), \tag{58}$$

which proves the result.