Series 7

Since we just started a new section, this exercise is short and simple!

I. Conformality condition in $d = 2$

1. *Check explicitly, that (2.1) is equivalent to (3.2) and (3.3).*

   We have to use the fact that on the plane $g^{\mu\nu} = \delta^{\mu\nu}$. So under a coordinate transformation $z^\mu \to w^\mu(x)$ the metric tensor transforms as:

$$g^{\mu\nu} \to \left(\frac{\partial w^\mu}{\partial z^\alpha}\right)\left(\frac{\partial w^\nu}{\partial z^\beta}\right) g^{\alpha\beta}. \tag{1}$$

   Hence,

$$\Lambda g^{\mu\nu} = \left(\frac{\partial w^\mu}{\partial z^\alpha}\right)\left(\frac{\partial w^\nu}{\partial z^\beta}\right) g^{\alpha\beta}. \tag{2}$$

   Since $g^{11} = g^{00} = 1$, $g^{01} = g^{10} = 0$, we have to calculate the following three cases:

   (a) The first case is for $\mu = \nu = 1$:

$$\begin{aligned}
\Lambda g^{11} = \Lambda &= \frac{\partial w^1}{\partial z^\alpha}\frac{\partial w^1}{\partial z^\beta} g^{\alpha\beta} \\
&= \frac{\partial w^1}{\partial z^0}\frac{\partial w^1}{\partial z^0} g^{00} + \frac{\partial w^1}{\partial z^1}\frac{\partial w^1}{\partial z^1} g^{11} + \frac{\partial w^1}{\partial z^0}\frac{\partial w^1}{\partial z^1} g^{01} + \frac{\partial w^1}{\partial z^1}\frac{\partial w^1}{\partial z^0} g^{10} \\
&= \frac{\partial w^1}{\partial z^0}\frac{\partial w^1}{\partial z^0} g^{00} + \frac{\partial w^1}{\partial z^1}\frac{\partial w^1}{\partial z^1} g^{11} \\
&= \frac{\partial w^1}{\partial z^0}\frac{\partial w^1}{\partial z^0} + \frac{\partial w^1}{\partial z^1}\frac{\partial w^1}{\partial z^1} \\
&= \left(\frac{\partial w^1}{\partial z^0}\right)^2 + \left(\frac{\partial w^1}{\partial z^1}\right)^2 = \Lambda.
\end{aligned} \tag{3}$$

   (b) The second case is for $\mu = \nu = 0$:

$$\begin{aligned}
\Lambda g^{00} = \Lambda &= \frac{\partial w^0}{\partial z^\alpha}\frac{\partial w^0}{\partial z^\beta} g^{\alpha\beta} \\
&= \frac{\partial w^0}{\partial z^0}\frac{\partial w^0}{\partial z^0} g^{00} + \frac{\partial w^0}{\partial z^1}\frac{\partial w^0}{\partial z^1} g^{11} + \frac{\partial w^0}{\partial z^0}\frac{\partial w^0}{\partial z^1} g^{01} + \frac{\partial w^0}{\partial z^1}\frac{\partial w^0}{\partial z^0} g^{10} \\
&= \frac{\partial w^0}{\partial z^0}\frac{\partial w^0}{\partial z^0} g^{00} + \frac{\partial w^0}{\partial z^1}\frac{\partial w^0}{\partial z^1} g^{11} \\
&= \frac{\partial w^0}{\partial z^0}\frac{\partial w^0}{\partial z^0} + \frac{\partial w^0}{\partial z^1}\frac{\partial w^0}{\partial z^1} \\
&= \left(\frac{\partial w^0}{\partial z^0}\right)^2 + \left(\frac{\partial w^0}{\partial z^1}\right)^2 = \Lambda.
\end{aligned} \tag{4}$$

(c) The third case is for either $\mu = 1, \nu = 0$ or $\mu = 0, \nu = 1$. Either one gives the same condition. Thus for $\mu = 0, \nu = 1$:

$$\begin{aligned}
\Lambda g^{01} = 0 &= \frac{\partial w^0}{\partial z^\alpha}\frac{\partial w^1}{\partial z^\beta} g^{\alpha\beta} \\
&= \frac{\partial w^0}{\partial z^0}\frac{\partial w^1}{\partial z^0} g^{00} + \frac{\partial w^0}{\partial z^1}\frac{\partial w^1}{\partial z^1} g^{11} + \frac{\partial w^0}{\partial z^0}\frac{\partial w^1}{\partial z^1} g^{01} + \frac{\partial w^0}{\partial z^1}\frac{\partial w^1}{\partial z^0} g^{10} \\
&= \frac{\partial w^0}{\partial z^0}\frac{\partial w^1}{\partial z^0} g^{00} + \frac{\partial w^0}{\partial z^1}\frac{\partial w^1}{\partial z^1} g^{11} \\
&= \frac{\partial w^0}{\partial z^0}\frac{\partial w^1}{\partial z^0} + \frac{\partial w^0}{\partial z^1}\frac{\partial w^1}{\partial z^1} = 0.
\end{aligned} \tag{5}$$

So in short, we can combine eq.(3) and eq.(4) into one, and the explicit condition that defines a conformal transformation is:

$$\left(\frac{\partial w^0}{\partial z^0}\right)^2 + \left(\frac{\partial w^0}{\partial z^1}\right)^2 = \left(\frac{\partial w^1}{\partial z^0}\right)^2 + \left(\frac{\partial w^1}{\partial z^1}\right)^2, \tag{6}$$

$$\frac{\partial w^0}{\partial z^0}\frac{\partial w^1}{\partial z^0} + \frac{\partial w^0}{\partial z^1}\frac{\partial w^1}{\partial z^1} = 0. \tag{7}$$

2. *Verify the equivalence of (3.2), (3.3) to (3.4) or (3.5).*

Starting from eq.(5) and assuming that none of the derivatives is zero, we have:

$$\frac{\partial w^0}{\partial z^0}\frac{\partial w^1}{\partial z^0} = -\frac{\partial w^0}{\partial z^1}\frac{\partial w^1}{\partial z^1}. \tag{8}$$

This is satisfied in the following four cases:

$$\frac{\partial w^0}{\partial z^0} = \frac{\partial w^0}{\partial z^1} \;\;\&\;\; \frac{\partial w^1}{\partial z^0} = -\frac{\partial w^1}{\partial z^1}, \tag{9}$$

$$\frac{\partial w^0}{\partial z^0} = -\frac{\partial w^0}{\partial z^1} \;\;\&\;\; \frac{\partial w^1}{\partial z^0} = \frac{\partial w^1}{\partial z^1}, \tag{10}$$

$$\frac{\partial w^0}{\partial z^0} = \frac{\partial w^1}{\partial z^1} \;\;\&\;\; \frac{\partial w^1}{\partial z^0} = -\frac{\partial w^0}{\partial z^1}, \tag{11}$$

$$\frac{\partial w^0}{\partial z^0} = -\frac{\partial w^1}{\partial z^1} \;\;\&\;\; \frac{\partial w^1}{\partial z^0} = \frac{\partial w^0}{\partial z^1}. \tag{12}$$

But now we should plug back these conditions one by one into eq.(6) and see for which of them the equality is satisfied. Then it is not hard to see that eq.(6) is satisfied only for:

$$\frac{\partial w^0}{\partial z^0} = \frac{\partial w^1}{\partial z^1} \;\;\&\;\; \frac{\partial w^1}{\partial z^0} = -\frac{\partial w^0}{\partial z^1}, \tag{13}$$

$$\frac{\partial w^0}{\partial z^0} = -\frac{\partial w^1}{\partial z^1} \;\;\&\;\; \frac{\partial w^1}{\partial z^0} = \frac{\partial w^0}{\partial z^1}. \tag{14}$$

Then eq.(13) is the Cauchy-Riemann equation for holomorphic functions, while eq.(14) defines antiholomorphic functions.

3. *Show that in complex coordinates, (3.4) can be written as (3.9).*

We start by writing $w(z,\bar{z}) = w^0 + iw^1$. Also, $\partial_{\bar{z}} = \frac{1}{2}(\partial_0 + i\partial_1)$. Then:

$$\begin{aligned}\partial_{\bar{z}} w(z,\bar{z}) = \frac{1}{2}(\partial_0 + i\partial_1)(w^0 + iw^1) = 0 \Rightarrow \\ \partial_0 w^0 + i\partial_0 w^1 + i\partial_1 w^0 - \partial_1 w^1 = 0 \Rightarrow \\ \left(\partial_0 w^0 - \partial_1 w^1\right) + i\left(\partial_0 w^1 + \partial_1 w^0\right) = 0.\end{aligned} \tag{15}$$

Hence, the following two conditions should apply simultaneously:

$$\begin{aligned}\partial_0 w^0 = \partial_1 w^1, \ \ \& \ \ \partial_0 w^1 = -\partial_1 w^0 \Rightarrow \\ \frac{\partial w^0}{\partial z^0} = \frac{\partial w^1}{\partial z^1}, \ \ \& \ \ \frac{\partial w^1}{\partial z^0} = -\frac{\partial w^0}{\partial z^1},\end{aligned} \tag{16}$$

which are the Cauchy-Riemann equations for holomorphic functions.

II. The group $SL(2,\mathbb{C})$

1. *Write down the explicit $SL(2,\mathbb{C})$ matrices corresponding to translations, rotations, dilations and SCT.*

For this question it is easier to use $f(z) = \frac{az+b}{cz+d}$ instead of the matrix notation. It is clear that these two methods are equivalent since for two matrices, A_1, A_2 we have:

$$A_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \qquad \& \qquad A_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}. \tag{17}$$

So,

$$\begin{aligned}A_1 A_2 &= \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \\ &= \begin{pmatrix} a_1 a_2 + b_1 c_2 & a_1 b_2 + b_1 d_2 \\ c_1 a_2 + d_1 c_2 & c_1 b_2 + d_1 d_2 \end{pmatrix}.\end{aligned} \tag{18}$$

For two functions, $f_1(z) = \frac{a_1 z + b_1}{c_1 z + d_1}, f_2(z) = \frac{a_2 z + b_2}{c_2 z + d_2}$ we have that:

$$\begin{aligned}f_1(z) \circ f_2(z) &= f_1(f_2(z)) \\ &= \frac{a_1\left(\frac{a_2 z + b_2}{c_2 z + d_2}\right) + b_1}{c_1\left(\frac{a_2 z + b_2}{c_2 z + d_2}\right) + d_1} \\ &= \frac{z\left(a_1 a_2 + b_1 c_2\right) + a_1 b_2 + b_1 d_2}{z\left(d_1 c_2 + c_1 a_2\right) + c_1 b_2 + d_1 d_2}.\end{aligned} \tag{19}$$

This is valid as long as $ad - bc = 1$.
Thus, it is easy to calculate the matrices that correspond to translations, rotations, dilatations and SCTs.

(a) For translations we have that $z' = z + \overline{a}$. So, $f(z) = z + \overline{a}$. Hence:

$$z + \overline{a} = \frac{az+b}{cz+d}, \quad ad - bc = 1. \tag{20}$$

It is easy to see that for $c = 0$ we have $ad = 1 \Rightarrow a = \frac{1}{d}$. So,

$$\frac{z+bd}{d^2} = z + \overline{a} \Rightarrow z + bd = d^2 z + d^2 \overline{a} \Rightarrow d = 1, \quad b = \overline{a}. \tag{21}$$

and the matrix is:

$$\mathcal{M}_{transl.} = \begin{pmatrix} 1 & \overline{a} \\ 0 & 1 \end{pmatrix}. \tag{22}$$

(b) Rotations in the complex plane for a vector $z = z_0 + iz_1$ is a map $z' \to ze^{i\theta}, \ \ \theta \in \mathbb{R}$. Hence, $f(z) = ze^{i\theta}$. Thus we have:

$$ze^{i\theta} = \frac{az+b}{cz+d}, \quad ad - bc = 1. \tag{23}$$

Again, it is clear that $c = 0$ and $a = \frac{1}{d}$. So,

$$ze^{i\theta} = \frac{z}{d^2} + \frac{b}{d}. \tag{24}$$

Matching the two sides, we have that $b = 0$ and $d^2 = e^{-i\theta} \Rightarrow d = e^{-i\frac{\theta}{2}}$. This gives us $a = e^{i\frac{\theta}{2}}$, and the matrix is:

$$\mathcal{M}_{rot} = \begin{pmatrix} e^{i\frac{\theta}{2}} & 0 \\ 0 & e^{-i\frac{\theta}{2}} \end{pmatrix} \tag{25}$$

and we can easily check that $\det(\mathcal{M}_{rot}) = 1$.

(c) For dilatations we have that $z' = \lambda z$ and so $f(z) = \lambda z$. This gives us

$$\lambda z = \frac{az+b}{cz+d}, \quad ad - bc = 1. \tag{26}$$

For $c = 0$ and $a = \frac{1}{d}$ we have:

$$\lambda z = \frac{z}{d^2} + \frac{b}{d}. \tag{27}$$

Matching the two sides, we have that $b = 0$ and $d^2 = \frac{1}{\lambda} \Rightarrow d = \lambda^{-1/2}$. This gives us $a = \lambda^{1/2}$, and the matrix is:

$$\mathcal{M}_{dil.} = \begin{pmatrix} \lambda^{1/2} & 0 \\ 0 & \lambda^{-1/2} \end{pmatrix}, \tag{28}$$

and again, we can easily check that $\det(\mathcal{M}_{dil.}) = 1$.

(d) The last one that we should check is SCTs. In this case the transformation is a map from $z' \to \frac{z}{\bar{b}z+1}$. In order to find this, one should start with the generator of SCT in $d = 2$ which is $l_1 = -\partial_w,\ z = -\frac{1}{w} \Rightarrow l_1 = -z^2\partial_z$. Since this is unnecessary for our case, we can just use that $f(z) = \frac{z}{\bar{b}z+1}$. Thus,

$$\frac{z}{\bar{b}z+1} = \frac{az+b}{cz+d}, \quad ad - bc = 1. \tag{29}$$

Matching the two sides we have:

$$z = az + b, \tag{30}$$

$$\bar{b}z + 1 = cz + d. \tag{31}$$

So clearly $a = 1$ and $b = 0$ and from $ad - bc = 1$ we have that $d = 1$. Hence, the only choice for $c = \bar{b}$. Thus the matrix is:

$$\mathcal{M}_{SCT} = \begin{pmatrix} 1 & 0 \\ \bar{b} & 1 \end{pmatrix}, \tag{32}$$

with a determinant of one.

2. *Given three points z_1, z_2, z_3, find the explicit $SL(2,\mathbb{C})$ transformation that maps them to 0, 1, ∞, respectively.*

- We shall start with $z_1 \to 0$. Then, assuming that z_1 is known, we have that:

$$\frac{az_1+b}{cz_1+d} = 0, \qquad ad - bc = 1. \tag{33}$$

This gives us that $az_1 + b = 0 \Rightarrow b = -az_1$. Also, it should be clear that in order to avoid infinities, that $cz_1 + d \neq 0$. Without loss of generality we can pick $c = 0$. Then since $ad = 1 \Rightarrow d = \frac{1}{a}$. Then the matrix is simply:

$$\mathcal{M} = \begin{pmatrix} a & -az_1 \\ 0 & \frac{1}{a} \end{pmatrix}, \tag{34}$$

written in terms of just one unknown function $a \in \mathbb{C}$. It is easy to check that $\det \mathcal{M} = 1$.
Indeed, without loss of generality we can pick $a = 1$ in which case the matrix takes the very simple form:

$$\mathcal{M}' = \begin{pmatrix} 1 & -z_1 \\ 0 & 1 \end{pmatrix} \tag{35}$$

We see that

$$f(z_1) = \frac{az_1+b}{cz_1+d} = \frac{z_1 - z_1}{1} = 0 \tag{36}$$

as requested.

- We then want to check explicitly the transformation $z_2 \to \infty$. In this case we have:

$$f(z_2) = \frac{az_2 + b}{cz_2 + d} \to \infty. \tag{37}$$

It is clear that

$$az_2 + b \neq \infty \ \ \& \ \ az_2 + b \neq 0 \tag{38}$$
$$cz_2 + d = 0 \Rightarrow d = -cz_2. \tag{39}$$

Then we can find that

$$\begin{aligned} ad - bc = 1 \Rightarrow a\big(-cz_2\big) - bc = 1 \\ \Rightarrow a = -\left(\frac{1+bc}{cz_2}\right), \ \ c \neq 0. \end{aligned} \tag{40}$$

Hence the matrix is:

$$\mathcal{M} = \begin{pmatrix} -\left(\frac{1+bc}{cz_2}\right) & b \\ c & -cz_2 \end{pmatrix}, \quad b \in \mathbb{C}, \ \ c \in \mathbb{C}^* \tag{41}$$

Indeed, $az_2 + b \neq 0$ as long as $z_2 \neq 0$ and also:

$$f(z_2) = \frac{az_2 + b}{cz_2 + d} = \frac{-\left(\frac{1+bc}{cz_2}\right)z_2 + b}{cz_2 - cz_2} = \frac{-1}{c(cz_2 - cz_2)} \to \infty. \tag{42}$$

- Finally we want the transformation $z_3 \to 1$. Hence, we have that:

$$f(z_3) = 1 \Rightarrow \frac{az_3 + b}{cz_3 + d} = 1 \Rightarrow az_3 + b = cz_3 + d. \tag{43}$$

Also from $ad - bc = 1$ we have:

$$ad - bc = 1 \Rightarrow a = \frac{bc+1}{d}, \ \ d \neq 0. \tag{44}$$

Then if we plug eq.(44) into eq.(43) we get:

$$\begin{aligned} &\frac{bc+1}{d} z_3 + b = cz_3 + d \Rightarrow \\ &bcz_3 + z_3 + bd = cdz_3 + d^2 \Rightarrow \\ &c(b-d)z_3 = d(d-b) - z_3 \Rightarrow \\ &c = \frac{d(d-b)}{(b-d)z_3} - \frac{z_3}{z_3(b-d)} \Rightarrow \\ &c = \frac{d^2 - bd - z_3}{(b-d)z_3}. \end{aligned} \tag{45}$$

Then we can find a since

$$\begin{aligned} a &= \frac{1+bc}{d} \\ &= \frac{1+b\left(\frac{d^2-bd-z_3}{(b-d)z_3}\right)}{d} \\ &= \frac{1}{d} + \frac{b}{d}\frac{(d^2-bd-z_3)}{(b-d)z_3}. \end{aligned} \tag{46}$$

And thus the matrix is:

$$\mathcal{M} = \begin{pmatrix} \left(\frac{1}{d} + \frac{b}{d}\frac{(d^2-bd-z_3)}{(b-d)z_3}\right) & b \\ \left(\frac{d^2-bd-z_3}{(b-d)z_3}\right) & d \end{pmatrix}, \quad b \in \mathbb{C}, \;\; d \in \mathbb{C}^*. \tag{47}$$

Indeed,

$$\begin{aligned} \det \mathcal{M} = ad - bc &= \left(\frac{1}{d} + \frac{b}{d}\frac{(d^2-bd-z_3)}{(b-d)z_3}\right)d - b\left(\frac{d^2-bd-z_3}{(b-d)z_3}\right) \\ &= 1 + b\frac{(d^2-bd-z_3)}{(b-d)z_3} - b\frac{(d^2-bd-z_3)}{(b-d)z_3} = 1. \end{aligned} \tag{48}$$

For simplicity and without loss of generality, we can take $b = 0$. Then:

$$\mathcal{M}' = \begin{pmatrix} \left(\frac{1}{d}\right) & 0 \\ \left(\frac{d^2-z_3}{-dz_3}\right) & d \end{pmatrix}, \;\; d \in \mathbb{C}^*. \tag{49}$$

Hence

$$f(z_3) = \frac{az_3+b}{cz_3+d} = \frac{\frac{z_3}{d}}{\frac{d^2-z_3}{-dz_3}z_3 + d} = \frac{\frac{z_3}{d}}{\frac{d^2z_3 - {z_3}^2 - d^2z_3}{-dz_3}} = \frac{-d{z_3}^2}{-d{z_3}^2} = 1, \tag{50}$$

as requested.