Series 8

I. Variation of a primary field in $d = 2$

1. *Verify (3.21) starting from (3.20) for an infinitesimal transformation*

$$w = z + \epsilon(z), \qquad \bar{w} = \bar{z} + \bar{\epsilon}(z). \tag{1}$$

We have the following:

$$\begin{aligned}
\delta_{\epsilon\bar{\epsilon}}\phi &= \phi^{'}(z,\bar{z}) - \phi(z,\bar{z}) \\
&= \phi^{'}\left(w - \epsilon(z), \overline{w} - \overline{\epsilon}(\bar{z})\right) - \phi(z,\bar{z}) \\
&= \phi^{'}(w,\overline{w}) - \epsilon(z)\partial_z\phi^{'}(w,\overline{w}) - \overline{\epsilon}(\bar{z})\partial_{\bar{z}}\phi^{'}(w,\overline{w}) - \phi(z,\bar{z}) \\
&= \Big[\Big(1 + \partial_z\epsilon(z)\Big)^{-h}\Big(1 + \partial_{\bar{z}}\overline{\epsilon}(\bar{z})\Big)^{-\overline{h}}\Big(1 - \epsilon(z)\partial_z - \overline{\epsilon}(\bar{z})\partial_{\bar{z}}\Big) - 1\Big] \times \phi(z,\bar{z}) \\
&= \Big[\Big(1 - h\partial_z\epsilon(z)\Big)\Big(1 - \overline{h}\partial_{\bar{z}}\overline{\epsilon}(\bar{z})\Big)\Big(1 - \epsilon(z)\partial_z - \overline{\epsilon}(\bar{z})\partial_{\bar{z}}\Big) - 1\Big] \times \phi(z,\bar{z}) \\
&= \Big[- \Big(h\partial_z\epsilon(z) + \epsilon(z)\partial_z\Big) - \Big(\overline{h}\partial_{\bar{z}}\overline{\epsilon}(\bar{z}) + \overline{\epsilon}(\bar{z})\partial_{\bar{z}}\Big) + \mathcal{O}(\epsilon^2, \overline{\epsilon}^2)\Big] \times \phi(z,\bar{z}) \\
&= -\Big(h\phi\partial_z\epsilon + \epsilon\partial_z\phi\Big) - \Big(\overline{h}\phi\partial_{\bar{z}}\overline{\epsilon} + \overline{\epsilon}\partial_{\bar{z}}\phi\Big).
\end{aligned} \tag{2}$$

Where we used the following:

- To go from the first line to the second, we rewrote $z = w - \epsilon(z)$ and $\bar{z} = \overline{w} - \overline{\epsilon}(\bar{z})$ since this enables us to use the Taylor expansion.
- To go from the second line to the third, we used the aforementioned expansion, which is:

$$f(x+h) = f(x) + hf'(x) + \dots \tag{3}$$

or more generally:

$$f(x_1 + h_1, x_2 + h_2) = f(x_1, x_2) + h_1\partial_{x_1}f(x_1,x_2) + h_2\partial_{x_2}f(x_1,x_2) + \dots \tag{4}$$

with $h_1 = -\epsilon(z)$ and $h_2 = -\overline{\epsilon}(\bar{z})$.
This step was crucial since we know how to calculate $\phi^{'}(w,\overline{w})$.
- To go from the third line to the fourth we used two things:
  (a) First we used that:

$$\phi^{'}(w,\overline{w}) = \left(\frac{dw}{dz}\right)^{-h}\left(\frac{d\overline{w}}{d\bar{z}}\right)^{-\overline{h}}\phi(z,\bar{z}). \tag{5}$$

But for $w = z + \epsilon(z)$ and $\overline{w} = \bar{z} + \overline{\epsilon}(\bar{z})$ we have:

$$\begin{aligned}
\phi^{'}(w,\overline{w}) &= \phi^{'}(z + \epsilon(z), \bar{z} + \overline{\epsilon}(\bar{z})) \\
&= \Big(1 + \partial_z\epsilon(z)\Big)^{-h}\Big(1 + \partial_{\bar{z}}\overline{\epsilon}(\bar{z})\Big)^{-\overline{h}} \times \phi(z,\bar{z}).
\end{aligned} \tag{6}$$

(b) Then we used that:

$$\begin{aligned}\partial_z\phi^{'}(w,\overline{w}) &= \partial_z\left(\left(1+\partial_z\epsilon(z)\right)^{-h}\left(1+\partial_{\overline{z}}\overline{\epsilon}(\overline{z})\right)^{-\overline{h}}\phi(z,\overline{z})\right)\\ &= \left(1+\partial_z\epsilon(z)\right)^{-h}\left(1+\partial_{\overline{z}}\overline{\epsilon}(\overline{z})\right)^{-\overline{h}}\times\partial_z\phi(z,\overline{z}),\end{aligned} \tag{7}$$

where we used that the derivatives of the first two terms vanish since $\partial_z\partial_z\epsilon(z) = \partial_z\partial_{\overline{z}}\overline{\epsilon}(\overline{z}) = 0$.
A similar result holds for $\partial_{\overline{z}}\phi^{'}(w,\overline{w})$

- To go from the fourth line to the fifth we used the identity

$$(1+c)^a = 1+ac, \quad c \ll 1. \tag{8}$$

- To go from the fifth line to the sixth we multiplied the products in the bracket, and we kept terms that were linear to $\epsilon(z)$ or $\partial_z\epsilon(z)$ and their antiholomorphic counterparts.
- In the seventh line we just redrafted the result in a more likeable fashion.

II. Correlation functions in $d = 2$

1. *Derive (3.23) from (2.28).*

We know that in d dimensions the two-point function is

$$\langle\phi_1(x_1)\phi_2(x_2)\rangle = \frac{1}{|x_1-x_2|^{\Delta_1+\Delta_2}} = \frac{1}{|x_1-x_2|^{2\Delta}}, \quad \Delta_1 = \Delta_2 = \Delta. \tag{9}$$

In $d = 2$ we can write x^μ in terms of $z, \overline{z}$. In this case we note the following:

(a) Firstly, the distance $|x_1 - x_2| = \sqrt{g_{\mu\nu}(x_1^\mu - x_2^\mu)(x_1^\nu - x_2^\nu)}$ now becomes:

$$\begin{aligned}|x_1-x_2| &= \sqrt{g_{z\overline{z}}(z_1-z_2)(\overline{z}_1-\overline{z}_2)+g_{\overline{z}z}(\overline{z}_1-\overline{z}_2)(z_1-z_2)}\\ &= \sqrt{\frac{1}{2}(z_1-z_2)(\overline{z}_1-\overline{z}_2)+\frac{1}{2}(\overline{z}_1-\overline{z}_2)(z_1-z_2)}\\ &= \sqrt{(z_1-z_2)(\overline{z}_1-\overline{z}_2)}\\ &= \left((z_1-z_2)(\overline{z}_1-\overline{z}_2)\right)^{1/2}\\ &= \left(z_{ij}\overline{z}_{ij}\right)^{1/2}, \quad i,j = 1,2.\end{aligned} \tag{10}$$

(b) Wer use the holomorphic and antiholomorphic conformal dimensions defined as

$$h = \frac{1}{2}(\Delta+s), \qquad \overline{h} = \frac{1}{2}(\Delta-s). \tag{11}$$

From this we can clearly see that:

$$\Delta = h+\overline{h}, \tag{12}$$

$$\Delta = 2h - s, \tag{13}$$

$$\Delta = 2\overline{h}+s. \tag{14}$$

Now, we want to compute the two-point function. Then:

$$\begin{aligned}\langle\phi_1(z_1,\bar{z}_1)\phi_2(z_2,\bar{z}_2)\rangle &= \frac{C_{12}}{\left(z_{ij}\bar{z}_{ij}\right)^{1/2^{2\Delta}}}\\ &= \frac{C_{12}}{\left(z_{ij}\bar{z}_{ij}\right)^{\Delta}}\\ &= \frac{C_{12}}{\left(z_{ij}\right)^{\Delta}\left(\bar{z}_{ij}\right)^{\Delta}}\\ &= \frac{C_{12}}{\left(z_{ij}\right)^{h+\bar{h}}\left(\bar{z}_{ij}\right)^{h+\bar{h}}}.\end{aligned} \tag{15}$$

But we can also see that:

$$\begin{aligned}\langle\phi_1(z_1,\bar{z}_1)\phi_2(z_2,\bar{z}_2)\rangle &= \frac{C_{12}}{\left(z_{ij}\right)^{\Delta}\left(\bar{z}_{ij}\right)^{\Delta}}\\ &= \frac{C_{12}}{\left(z_{ij}\right)^{2h-s}\left(\bar{z}_{ij}\right)^{2\bar{h}+s}}.\end{aligned} \tag{16}$$

If we equate eq.(15) and eq.(16) we get the following conditions:

$$h+\bar{h} = 2h-s, \qquad h+\bar{h} = 2\bar{h}+s, \tag{17}$$

which lead to

$$h-\bar{h} = s. \tag{18}$$

But from rotation invariance we know that the sum of the spin in the correlator should be zero, i.e. $s=0$, which leads to $h=\bar{h}$. So the two-point function can be written as:

$$\langle\phi_1(z_1,\bar{z}_1)\phi_2(z_2,\bar{z}_2)\rangle = \frac{C_{12}}{\left(z_{ij}\right)^{2h}\left(\bar{z}_{ij}\right)^{2\bar{h}}}. \tag{19}$$

2. *Derive (3.24) from (2.32).*

For the three-point function in d-dimensions we have that:

$$\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \frac{C_{123}}{x_{12}{}^{\Delta_1+\Delta_2-\Delta_3}x_{23}{}^{\Delta_2+\Delta_3-\Delta_1}x_{13}{}^{\Delta_1+\Delta_3-\Delta_2}}. \tag{20}$$

If we again express it in terms of $z,\bar{z}$ in two dimensions, we shall use eq.(10) to express the distance. Then for x_{12}:

$$\begin{aligned}\left(z_{12}\bar{z}_{12}\right)^{\frac{\Delta_1+\Delta_2-\Delta_3}{2}} &= \left(z_{12}\right)^{\frac{\Delta_1+\Delta_2-\Delta_3}{2}}\left(\bar{z}_{12}\right)^{\frac{\Delta_1+\Delta_2-\Delta_3}{2}}\\ &= \left(z_{12}\right)^{h_1+h_2-h_3-\frac{s_1+s_2-s_3}{2}}\left(\bar{z}_{12}\right)^{\bar{h}_1+\bar{h}_2-\bar{h}_3+\frac{s_1+s_2-s_3}{2}}\\ &= \left(z_{12}\right)^{\frac{h_1+h_2-h_3+\bar{h}_1+\bar{h}_2-\bar{h}_3}{2}}\left(\bar{z}_{12}\right)^{\frac{\bar{h}_1+\bar{h}_2-\bar{h}_3+h_1+h_2-h_3}{2}}.\end{aligned} \tag{21}$$

And since the sum of the spins from the holomorphic part should cancel the sum of the spins from the antiholomorphic part, the only way for the equality to stand is for

$$h_1 = \bar{h}_1, \tag{22}$$
$$h_2 = \bar{h}_2, \tag{23}$$
$$h_3 = \bar{h}_3. \tag{24}$$

Since this is the same for x_{23} and x_{13} parts, the three-point function can be written as:

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = C_{123} \frac{1}{z_{12}^{h_1+h_2-h_3} z_{23}^{h_2+h_3-h_1} z_{13}^{h_1+h_3-h_2}} \times \frac{1}{\bar{z}_{12}^{\bar{h}_1+\bar{h}_2-\bar{h}_3} \bar{z}_{23}^{\bar{h}_2+\bar{h}_3-\bar{h}_1} \bar{z}_{13}^{\bar{h}_1+\bar{h}_3-\bar{h}_2}}. \tag{25}$$

3. *Check that (3.28) and (3.29) are equivalent to (3.27).*

We are going to use the following two identities to verify the desired equivalences:

$$\eta = \frac{z_{12}z_{34}}{z_{13}z_{24}}, \tag{26}$$
$$1-\eta = \frac{z_{14}z_{23}}{z_{13}z_{24}}. \tag{27}$$

Then we are asked to verify that:

$$\begin{aligned} \left[z_{12}z_{13}z_{14}z_{23}z_{24}z_{34}\right]^{-2h/3} &= \frac{(1-\eta)^{4h/3}}{\eta^{2h/3}} \frac{1}{(z_{14}z_{23})^{2h}} \\ &= \left[\eta(1-\eta)\right]^{4h/3} \left(\frac{z_{13}z_{24}}{z_{12}z_{23}z_{14}z_{34}}\right)^{2h}. \end{aligned} \tag{28}$$

- For the first equality we have:

$$\begin{aligned} &\frac{(1-\eta)^{4h/3}}{\eta^{2h/3}} \frac{1}{(z_{14}z_{23})^{2h}} = \frac{\left(\frac{z_{14}z_{23}}{z_{13}z_{24}}\right)^{4h/3}}{\left(\frac{z_{12}z_{34}}{z_{13}z_{24}}\right)^{2h/3}} \frac{1}{(z_{14}z_{23})^{2h}} = \\ &= \frac{z_{14}{}^{4h/3} z_{23}{}^{4h/3} z_{13}{}^{-4h/3} z_{24}{}^{-4h/3}}{z_{12}{}^{2h/3} z_{34}{}^{2h/3} z_{13}{}^{-2h/3} z_{24}{}^{-2h/3} z_{14}{}^{2h} z_{23}{}^{2h}} \\ &= z_{12}{}^{(-2h/3)} z_{13}{}^{(-4h/3+2h/3)} z_{14}{}^{(4h/3-2h)} z_{23}{}^{(4h/3-2h)} z_{24}{}^{(-4h/3+2h/3)} z_{34}{}^{(-2h/3)} \\ &= \left[z_{12}z_{13}z_{14}z_{23}z_{24}z_{34}\right]^{-2h/3}. \end{aligned} \tag{29}$$

- For the second equality we have:

$$
\begin{aligned}
&\Big[\eta(1-\eta)\Big]^{4h/3}\left(\frac{z_{13}z_{24}}{z_{12}z_{23}z_{14}z_{34}}\right)^{2h} = \\
&= \left(\frac{z_{12}z_{34}}{z_{13}z_{24}}\right)^{4h/3}\left(\frac{z_{14}z_{23}}{z_{13}z_{24}}\right)^{4h/3}\left(\frac{z_{13}z_{24}}{z_{12}z_{23}z_{14}z_{34}}\right)^{2h} = \\
&= z_{12}{}^{(4h/3-2h)}z_{13}{}^{(-8h/3+2h)}z_{14}{}^{(4h/3-2h)}z_{23}{}^{(4h/3-2h)}z_{24}{}^{(-8h/3+2h)}z_{34}{}^{(4h/3-2h)} \\
&= \Big[z_{12}z_{13}z_{14}z_{23}z_{24}z_{34}\Big]^{-2h/3}.
\end{aligned}
\tag{30}
$$

III. Ward identities in $d = 2$

1. *Derive (3.35a-d) from (3.30a-c).*

We are going to use the following equations:

$$\frac{\partial}{\partial x^{\mu}}\langle T^{\mu}{}_{\nu}(x)X\rangle = -\sum_{i=1}^{n}\delta(x-x_i)\frac{\partial}{\partial x_i^{\nu}}\langle X\rangle, \tag{31}$$

$$\epsilon_{\mu\nu}\langle T^{\mu\nu}(x)X\rangle = -i\sum_{i=1}^{n}\delta(x-x_i)s_i\langle X\rangle, \tag{32}$$

$$\langle T^{\mu}{}_{\mu}(x)X\rangle = -\sum_{i=1}^{n}\delta(x-x_i)\Delta_i\langle X\rangle, \tag{33}$$

along with

$$g^{\mu\nu} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \quad \& \quad \epsilon_{\mu\nu} = \begin{pmatrix} 0 & i/2 \\ -i/2 & 0 \end{pmatrix}. \tag{34}$$

(a) Starting with eq.(33) we rewrite $T^{\mu}{}_{\mu} = g^{\mu\nu}T_{\nu\mu}$. Then

$$
\begin{aligned}
&\langle g^{\mu\nu}T_{\nu\mu}X\rangle = -\sum_{i=1}^{n}\delta(x-x_i)\Delta_i\langle X\rangle \Rightarrow \\
&\langle\big(g^{z\bar{z}}T_{\bar{z}z} + g^{\bar{z}z}T_{z\bar{z}}\big)X\rangle = -\sum_{i=1}^{n}\delta(x-x_i)\Delta_i\langle X\rangle \Rightarrow \\
&2\langle\big(T_{\bar{z}z} + T_{z\bar{z}}\big)X\rangle = -\sum_{i=1}^{n}\delta(x-x_i)\Delta_i\langle X\rangle \Rightarrow \\
&2\langle T_{z\bar{z}}X\rangle + 2\langle T_{\bar{z}z}X\rangle = -\sum_{i=1}^{n}\delta(x-x_i)\Delta_i\langle X\rangle,
\end{aligned}
\tag{35}
$$

which verifies the result, and to go from the second to the third line we used that $g^{z\bar{z}} = g^{\bar{z}z} = 2$.

(b) From eq.(32) we write $T^{\mu\nu} = g^{\mu\rho}g^{\nu\sigma}T_{\rho\sigma}$ and then we have:

$$\begin{aligned}
&\epsilon_{\mu\nu}\langle g^{\mu\rho}g^{\nu\sigma}T_{\rho\sigma}X\rangle = -i\sum_{i=1}^{n}\delta(x-x_i)s_i\langle X\rangle \Rightarrow \\
&\epsilon_{z\bar{z}}\langle g^{z\rho}g^{\bar{z}\sigma}T_{\rho\sigma}X\rangle + \epsilon_{\bar{z}z}\langle g^{\bar{z}\rho}g^{z\sigma}T_{\rho\sigma}X\rangle = -i\sum_{i=1}^{n}\delta(x-x_i)s_i\langle X\rangle \Rightarrow \\
&\epsilon_{z\bar{z}}\langle g^{z\bar{z}}g^{\bar{z}z}T_{\bar{z}z}X\rangle + \epsilon_{\bar{z}z}\langle g^{\bar{z}z}g^{z\bar{z}}T_{z\bar{z}}X\rangle = -i\sum_{i=1}^{n}\delta(x-x_i)s_i\langle X\rangle \Rightarrow \\
&\frac{i}{2}\langle 4T_{\bar{z}z}X\rangle - \frac{i}{2}\langle 4T_{z\bar{z}}X\rangle = -i\sum_{i=1}^{n}\delta(x-x_i)s_i\langle X\rangle \Rightarrow \\
&2i\langle T_{\bar{z}z}X\rangle - 2i\langle T_{z\bar{z}}X\rangle = -i\sum_{i=1}^{n}\delta(x-x_i)s_i\langle X\rangle \Rightarrow \\
&2\langle T_{\bar{z}z}X\rangle - 2\langle T_{z\bar{z}}X\rangle = -\sum_{i=1}^{n}\delta(x-x_i)s_i\langle X\rangle,
\end{aligned} \tag{36}$$

which proves the result.

(c) The last two equations are derived from eq.(31). To prove these relations we shall also use the following identity (and its generalisation) which is very nicely proven in Di Francesco:

$$\delta(x) = \frac{1}{\pi}\partial_{\bar{z}}\frac{1}{z} = \frac{1}{\pi}\partial_z\frac{1}{\bar{z}}. \tag{37}$$

$$\delta(x-x_i) = \frac{1}{\pi}\partial_{\bar{z}}\frac{1}{z-z_i} = \frac{1}{\pi}\partial_z\frac{1}{\bar{z}-\bar{z}_i}. \tag{38}$$

The first equality is more useful when we are dealing with holomorphic functions, while the later is more useful for antiholomorphic.
The extra things that we should is is to write $T^{\mu}{}_{\nu} = g^{\mu\rho}T_{\rho\nu}$. Also for simplicity we rephrase $\frac{\partial}{\partial x^\mu} = \partial_{x^\mu}$ and $\frac{\partial}{\partial x_i^\nu} = \partial_{x_i^\nu}$. Then:

$$\begin{aligned}
&\partial_{x^\mu}\langle g^{\mu\rho}T_{\rho\nu}X\rangle = -\sum_{i=1}^{n}\delta(x-x_i)\partial_{x_i^\nu}\langle X\rangle \Rightarrow \\
&\partial_z\langle g^{z\rho}T_{\rho\nu}X\rangle + \partial_{\bar{z}}\langle g^{\bar{z}\rho}T_{\rho\nu}X\rangle = -\sum_{i=1}^{n}\delta(x-x_i)\partial_{x_i^\nu}\langle X\rangle \Rightarrow \\
&\partial_z\langle g^{z\bar{z}}T_{\bar{z}\nu}X\rangle + \partial_{\bar{z}}\langle g^{\bar{z}z}T_{z\nu}X\rangle = -\sum_{i=1}^{n}\delta(x-x_i)\partial_{x_i^\nu}\langle X\rangle \Rightarrow \\
&2\partial_z\langle T_{\bar{z}\nu}X\rangle + 2\partial_{\bar{z}}\langle T_{z\nu}X\rangle = -\sum_{i=1}^{n}\delta(x-x_i)\partial_{x_i^\nu}\langle X\rangle.
\end{aligned} \tag{39}$$

Now we have a choice to make. We should take $\nu = z = w$ or $\nu = \bar{z} = \bar{w}$.

i. For $\nu = z = w$ we have the following:

$$\begin{aligned}
2\partial_z\langle T_{\bar{z}z}X\rangle + 2\partial_{\bar{z}}\langle T_{zz}X\rangle &= -\sum_{i=1}^{n}\delta(x-x_i)\partial_{w_i}\langle X\rangle \Rightarrow \\
2\partial_z\langle T_{\bar{z}z}X\rangle + 2\partial_{\bar{z}}\langle T_{zz}X\rangle &= -\sum_{i=1}^{n}\frac{1}{\pi}\partial_{\bar{z}}\frac{1}{z-w_i}\partial_{w_i}\langle X\rangle \Rightarrow \\
2\pi\partial_z\langle T_{\bar{z}z}X\rangle + 2\pi\partial_{\bar{z}}\langle T_{zz}X\rangle &= -\sum_{i=1}^{n}\partial_{\bar{z}}\frac{1}{z-w_i}\partial_{w_i}\langle X\rangle.
\end{aligned} \tag{40}$$

ii. For $\nu = \bar{z} = \bar{w}$ we have that:

$$\begin{aligned}
2\partial_z\langle T_{\bar{z}\bar{z}}X\rangle + 2\partial_{\bar{z}}\langle T_{z\bar{z}}X\rangle &= -\sum_{i=1}^{n}\delta(x-x_i)\partial_{\bar{w}_i}\langle X\rangle \Rightarrow \\
2\partial_z\langle T_{\bar{z}\bar{z}}X\rangle + 2\partial_{\bar{z}}\langle T_{z\bar{z}}X\rangle &= -\sum_{i=1}^{n}\frac{1}{\pi}\partial_{z}\frac{1}{\bar{z}-\bar{w}_i}\partial_{\bar{w}_i}\langle X\rangle \Rightarrow \\
2\pi\partial_z\langle T_{\bar{z}\bar{z}}X\rangle + 2\pi\partial_{\bar{z}}\langle T_{z\bar{z}}X\rangle &= -\sum_{i=1}^{n}\partial_{z}\frac{1}{\bar{z}-\bar{w}_i}\partial_{\bar{w}_i}\langle X\rangle.
\end{aligned} \tag{41}$$

Thus at the end we have the following four Ward identities:

$$2\pi\partial_z\langle T_{\bar{z}z}X\rangle + 2\pi\partial_{\bar{z}}\langle T_{zz}X\rangle = -\sum_{i=1}^{n}\partial_{\bar{z}}\frac{1}{z-w_i}\partial_{w_i}\langle X\rangle \tag{42}$$

$$2\pi\partial_z\langle T_{\bar{z}\bar{z}}X\rangle + 2\pi\partial_{\bar{z}}\langle T_{z\bar{z}}X\rangle = -\sum_{i=1}^{n}\partial_{z}\frac{1}{\bar{z}-\bar{w}_i}\partial_{\bar{w}_i}\langle X\rangle \tag{43}$$

$$2\langle T_{z\bar{z}}X\rangle + 2\langle T_{\bar{z}z}X\rangle = -\sum_{i=1}^{n}\delta(x-x_i)\Delta_i\langle X\rangle \tag{44}$$

$$2\langle T_{\bar{z}z}X\rangle - 2\langle T_{z\bar{z}}X\rangle = -\sum_{i=1}^{n}\delta(x-x_i)s_i\langle X\rangle. \tag{45}$$

2. *Verify (3.36a,b).*

- If we add eq.(44) and eq.(45) we get that:

$$\begin{aligned}
4\langle T_{\bar{z}z}X\rangle &= -\sum_{i=1}^{n}\delta(x-x_i)\big(\Delta_i+s_i\big)\langle X\rangle \Rightarrow \\
4\langle T_{\bar{z}z}X\rangle &= -\sum_{i=1}^{n}\delta(x-x_i)2h_i\langle X\rangle \Rightarrow \\
4\langle T_{\bar{z}z}X\rangle &= -\sum_{i=1}^{n}\frac{1}{\pi}\partial_{\bar{z}}\frac{1}{z-w_i}2h_i\langle X\rangle \Rightarrow \\
2\pi\langle T_{\bar{z}z}X\rangle &= -\sum_{i=1}^{n}\partial_{\bar{z}}\frac{1}{z-w_i}h_i\langle X\rangle,
\end{aligned} \tag{46}$$

where in the second line we used that $h = \frac{1}{2}(\Delta + s)$ and in the third line we used the identity of the delta function adapted for holomorphic functions since h_i is holomorphic.

- If we subtract eq.(45) and eq.(44) we get that:

$$\begin{aligned} 4\langle T_{z\bar{z}}X\rangle &= -\sum_{i=1}^{n} \delta(x - x_i)\big(\Delta_i - s_i\big)\langle X\rangle \Rightarrow \\ 4\langle T_{z\bar{z}}X\rangle &= -\sum_{i=1}^{n} \delta(x - x_i)2\bar{h}_i\langle X\rangle \Rightarrow \\ 4\langle T_{z\bar{z}}X\rangle &= -\sum_{i=1}^{n} \frac{1}{\pi}\partial_z \frac{1}{\bar{z} - \bar{w}_i}2\bar{h}_i\langle X\rangle \Rightarrow \\ 2\pi\langle T_{z\bar{z}}X\rangle &= -\sum_{i=1}^{n} \partial_z \frac{1}{\bar{z} - \bar{w}_i}\bar{h}_i\langle X\rangle. \end{aligned} \tag{47}$$

where in the second line we used that $\bar{h} = \frac{1}{2}(\Delta - s)$ and in the third line we used the identity of the delta function adapted for antiholomorphic functions since $\bar{h}_i$ is antiholomorphic.

3. *Verify (3.37a,b).*

We want to verify eq(3.37a) and eq.(3.37b) of the notes.

- For the first one, what we should do is to insert eq.(46) into eq.(42). Thus:

$$\begin{aligned}
&2\pi\partial_z\langle T_{\bar{z}z}X\rangle + 2\pi\partial_{\bar{z}}\langle T_{zz}X\rangle = -\sum_{i=1}^{n}\partial_{\bar{z}}\frac{1}{z - w_i}\partial_{w_i}\langle X\rangle \Rightarrow \\
&\partial_z\Big(2\pi\langle T_{\bar{z}z}X\rangle\Big) + \partial_{\bar{z}}\Big(2\pi\langle T_{zz}X\rangle\Big) + \sum_{i=1}^{n}\partial_{\bar{z}}\frac{1}{z - w_i}\partial_{w_i}\langle X\rangle = 0 \Rightarrow \\
&\partial_z\Big(-\sum_{i=1}^{n}\partial_{\bar{z}}\frac{1}{z - w_i}h_i\langle X\rangle\Big) + \partial_{\bar{z}}\Big(2\pi\langle T_{zz}X\rangle\Big) + \sum_{i=1}^{n}\partial_{\bar{z}}\frac{1}{z - w_i}\partial_{w_i}\langle X\rangle = 0 \Rightarrow \\
&\partial_{\bar{z}}\Big\{\partial_z\Big(-\sum_{i=1}^{n}\frac{1}{z - w_i}h_i\langle X\rangle\Big) + \langle 2\pi T_{zz}X\rangle + \sum_{i=1}^{n}\frac{1}{z - w_i}\partial_{w_i}\langle X\rangle\Big\} = 0 \Rightarrow \\
&\partial_{\bar{z}}\Big\{-\sum_{i=1}^{n}\partial_z\Big(\frac{1}{z - w_i}h_i\langle X\rangle\Big) + \sum_{i=1}^{n}\frac{1}{z - w_i}\partial_{w_i}\langle X\rangle - \langle T(z,\bar{z})X\rangle\Big\} = 0 \Rightarrow \\
&\partial_{\bar{z}}\Big\{+\sum_{i=1}^{n}\partial_z\Big(\frac{1}{z - w_i}h_i\langle X\rangle\Big) - \sum_{i=1}^{n}\frac{1}{z - w_i}\partial_{w_i}\langle X\rangle + \langle T(z,\bar{z})X\rangle\Big\} = 0 \Rightarrow \\
&\partial_{\bar{z}}\Big\{-\sum_{i=1}^{n}\Big[-\partial_z\Big(\frac{1}{z - w_i}h_i\langle X\rangle\Big) + \frac{1}{z - w_i}\partial_{w_i}\langle X\rangle\Big] + \langle T(z,\bar{z})X\rangle\Big\} = 0 \Rightarrow \\
&\partial_{\bar{z}}\Big\{-\sum_{i=1}^{n}\Big[\frac{1}{(z - w_i)^2}h_i\langle X\rangle + \frac{1}{z - w_i}\partial_{w_i}\langle X\rangle\Big] + \langle T(z,\bar{z})X\rangle\Big\} = 0 \Rightarrow \\
&\partial_{\bar{z}}\Big\{\langle T(z,\bar{z})X\rangle - \sum_{i=1}^{n}\Big[\frac{1}{z - w_i}\partial_{w_i}\langle X\rangle + \frac{h_i}{(z - w_i)^2}\langle X\rangle\Big]\Big\} = 0.
\end{aligned} \tag{48}$$

(a) In the first line we used eq.(42).
(b) In the second line we plugged the 2π inside the derivative as this does not change the outcome.
(c) In the third line we inserted eq.(46).
(d) In the forth line we used the fact that $\partial_z\partial_{\bar{z}} = \partial_{\bar{z}}\partial_z$ and we wrote the equation in terms of the common factor $\partial_{\bar{z}}$.
(e) In the fifth line we used that the renormalized energy-momentum tensor is

$$T = -2\pi T_{zz}. \tag{49}$$

(f) In the sixth line we just changed the signs.
(g) In the seventh line we combined the two sums into one.
(h) In the eighth line we used the fact that $\partial_x \frac{1}{x-a} = -\frac{1}{(x-a)^2}$

- For the second relation the procedure is exactly the same as for the first, but now the starting point is eq.(43) into which we insert eq.(47). The new renormalized energy-momentum tensor is $\overline{T} = -2\pi T_{\bar{z}\bar{z}}$.
Indeed :

$$\begin{aligned}
&2\pi\partial_z\langle T_{\bar{z}\bar{z}}X\rangle + 2\pi\partial_{\bar{z}}\langle T_{z\bar{z}}X\rangle = -\sum_{i=1}^{n}\partial_z\frac{1}{\bar{z}-\overline{w}_i}\partial_{\overline{w}_i}\langle X\rangle \Rightarrow\\
&\partial_z\Big(2\pi\langle T_{\bar{z}\bar{z}}X\rangle\Big) + \partial_{\bar{z}}\Big(2\pi\langle T_{z\bar{z}}X\rangle\Big) + \sum_{i=1}^{n}\partial_z\frac{1}{\bar{z}-\overline{w}_i}\partial_{\overline{w}_i}\langle X\rangle = 0 \Rightarrow\\
&\partial_z\Big(2\pi\langle T_{\bar{z}\bar{z}}X\rangle\Big) + \partial_{\bar{z}}\Big(-\sum_{i=1}^{n}\partial_z\frac{1}{\bar{z}-\overline{w}_i}\bar{h}_i\langle X\rangle\Big) + \sum_{i=1}^{n}\partial_z\frac{1}{\bar{z}-\overline{w}_i}\partial_{\overline{w}_i}\langle X\rangle = 0 \Rightarrow\\
&\partial_z\Big\{\langle 2\pi T_{\bar{z}\bar{z}}X\rangle + \partial_{\bar{z}}\Big(-\sum_{i=1}^{n}\frac{1}{\bar{z}-\overline{w}_i}\bar{h}_i\langle X\rangle\Big) + \sum_{i=1}^{n}\frac{1}{\bar{z}-\overline{w}_i}\partial_{\overline{w}_i}\langle X\rangle\Big\} = 0 \Rightarrow\\
&\partial_z\Big\{-\langle T(z\bar{z})X\rangle + \sum_{i=1}^{n}\Big[-\partial_{\bar{z}}\Big(\frac{1}{\bar{z}-\overline{w}_i}\bar{h}_i\langle X\rangle\Big) + \frac{1}{\bar{z}-\overline{w}_i}\partial_{\overline{w}_i}\langle X\rangle\Big]\Big\} = 0 \Rightarrow\\
&\partial_z\Big\{-\langle T(z\bar{z})X\rangle + \sum_{i=1}^{n}\Big[\frac{\bar{h}_i}{(\bar{z}-\overline{w}_i)^2}\langle X\rangle + \frac{1}{\bar{z}-\overline{w}_i}\partial_{\overline{w}_i}\langle X\rangle\Big]\Big\} = 0 \Rightarrow\\
&\partial_z\Big\{\langle T(z\bar{z})X\rangle - \sum_{i=1}^{n}\Big[\frac{\bar{h}_i}{(\bar{z}-\overline{w}_i)^2}\langle X\rangle + \frac{1}{\bar{z}-\overline{w}_i}\partial_{\overline{w}_i}\langle X\rangle\Big]\Big\} = 0.
\end{aligned} \tag{50}$$