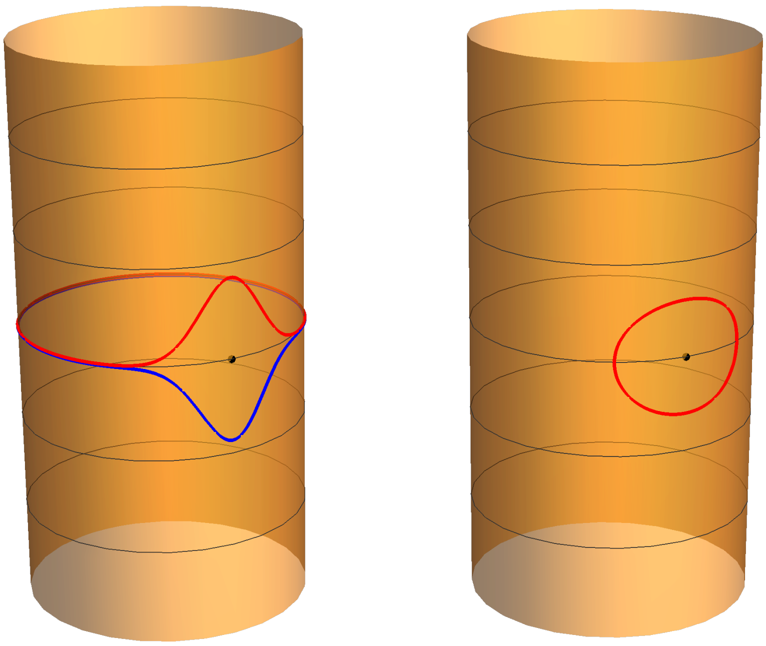
Series 9

I. Integration contours

Pull the figure of the integration contours for (3.53) on the z-plane back to the cylinder and visualize them in terms of equal-time contours.



II. Sort-distance singularities of T_{zz}, $T_{\bar{z}\bar{z}}$ *with* Φ

Derive (3.56) from (3.55) using residues.

We start by using the following:

$$-\frac{1}{2\pi i}\oint \mathrm{d}z\,\epsilon(z)\mathcal{R}\Big(T_{zz}\Phi(w,\bar{w})\Big) - \frac{1}{2\pi i}\oint d\bar{z}\,\bar{\epsilon}(\bar{z})\mathcal{R}\Big(T_{\bar{z}\bar{z}}\Phi(w,\bar{w})\Big) =$$
$$= -h\partial_w\epsilon(w)\Phi(w,\bar{w}) - \epsilon(w)\partial_w\Phi(w,\bar{w}) - \bar{h}\partial_{\bar{w}}\bar{\epsilon}(\bar{w})\Phi(w,\bar{w}) - \bar{\epsilon}(\bar{w})\partial_{\bar{w}}\Phi(w,\bar{w}). \tag{1}$$

We will explicitly solve for the first part, since the method for the second part is identical.

Any meromorphic function can be written as a Laurent series around the pole at $z = z_0$, i.e.:

$$\begin{aligned} f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots \\ + b_1(z-z_0)^{-1} + b_2(z-z_0)^{-2} + \dots \end{aligned} \tag{2}$$

Hence,

$$\mathcal{R}\Big(T_{zz}\Phi(w,\bar{w})\Big) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + b_1(z-z_0)^{-1} + b_2(z-z_0)^{-2} + \mathcal{O}(a_3, b_3), \tag{3}$$

so that:

$$\begin{aligned} &-\frac{1}{2\pi i}\oint \mathrm{d}z\epsilon(z)\Big\{a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + b_1(z-z_0)^{-1} + b_2(z-z_0)^{-2}\Big\} = \\ &= -\frac{1}{2\pi i} \times 2\pi i \times \Big(\text{Sum of residues}\Big). \end{aligned} \tag{4}$$

So our task is to calculate the residues. In general for a residue c_i of rank m we have:

$$c_i = \frac{1}{(m-1)!}\lim_{z\to z_0}\left\{\frac{\partial^{m-1}}{\partial z^{m-1}}\Big[\big(z-z_0\big)^m f(z)\Big]\right\}. \tag{5}$$

We take that $\epsilon(z)$ has no poles. Thus:

- $f_1(z) = a_o\epsilon(z)$ has no poles, so $Res(f_1(z)) = 0$.
- $f_2(z) = a_1(z-w)\epsilon(z)$ has a single pole at $z = w$ of rank $m = 1$:

$$c_1 = \lim_{z\to w}\Big\{(z-w)\epsilon(z)a_1(z-w)\Big\} = a_1\lim_{z\to w}\epsilon(z)(z-w)^2 = 0 \tag{6}$$

- Same is for any higher order of a_i, so we can neglect these terms.
- $f_3(z) = b_1\epsilon(z)(z-w)^{-1}$ has a pole at $z = w$ of rank $m = 1$. Thus:

$$c_2 = \lim_{z\to w}\left\{(z-w)\epsilon(z)\frac{b_1}{(z-w)}\right\} = b_1\epsilon(w) \tag{7}$$

- $f_4(z) = b_2\epsilon(z)(z-w)^{-2}$ has a pole at $z = w$ of rank $m = 2$. Hence:

$$\begin{aligned} c_3 = \frac{1}{(2-1)!}\lim_{z\to w}\left\{\frac{\partial}{\partial z}\Big[\big(z-w\big)^2 b_2\epsilon(z)(z-w)^{-2}\Big]\right\} &= b_2\lim_{z\to w}\Big[\frac{\partial}{\partial z}\epsilon(z)\Big] \\ &= b_2\partial_w\epsilon(w). \end{aligned} \tag{8}$$

So we have:

$$\begin{aligned} -\frac{1}{2\pi i}\oint dz\epsilon(z)\mathcal{R}\Big(T_{zz}\Phi(w,\bar{w})\Big) &= -\Big(b_2\partial_w\epsilon(w) + b_1\epsilon(w)\Big) \\ &= -\Big(h\partial_w\epsilon(w)\Phi(w,\bar{w}) + \epsilon(w)\partial_w\Phi(w,\bar{w})\Big). \end{aligned} \tag{9}$$

It is obvious that :

$$b_1 = \partial_w \Phi(w,\bar{w}), \qquad\qquad b_2 = h\Phi(w,\bar{w}) \tag{10}$$

and finally:

$$\mathcal{R}\Big(T_{zz}\Phi(w,\bar{w})\Big) = \frac{\partial_w \Phi(w,\bar{w})}{(z-w)} + \frac{h\Phi(w,\bar{w})}{(z-w)^2}. \tag{11}$$

The calculation for the antiholomorphic part works analogously.

III. The Schwartzian derivative

Calculate the Schwartzian derivative $S(f,z)$ for $f(z) = \frac{az+b}{cz+d}$ with $ad - bc = 1$.

We have to evaluate $S(f,z) = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2$ for $f(z) = \frac{az+b}{cz+d}$ with $ad - bc = 1$. We should calculate the following derivatives:

$$\partial_z f(z) = \frac{ad-bc}{(cz+d)^2} = \frac{1}{(cz+d)^2}, \tag{12}$$

$$\partial_z^2 f(z) = \frac{-2c(ad-bc)}{(cz+d)^3} = \frac{-2c}{(cz+d)^3}, \tag{13}$$

$$\partial_z^3 f(z) = \frac{6c^2(ad-bc)}{(cz+d)^4} = \frac{6c^2}{(cz+d)^4}. \tag{14}$$

Thus:

$$S(f,z) = \frac{\frac{1}{(cz+d)^2} \times \frac{6c^2}{(cz+d)^4} - \frac{3}{2}\left(\frac{-2c}{(cz+d)^3}\right)^2}{\left(\frac{1}{(cz+d)^2}\right)^2} = \frac{\frac{6c^2}{(cz+d)^6} - \frac{3}{2}\left(\frac{4c^2}{(cz+d)^6}\right)}{\left(\frac{1}{(cz+d)^2}\right)^2} = 0, \tag{15}$$

which was the expected result, as an $SL(2,\mathbb{C})$ transformation has zero Schwartzian derivative.