Series 10

I. Mode expansion of T_{zz}: calculate $\langle T_{zz}(z)T_{ww}(w)\rangle$

Verify equation (3.85) in the notes.

We need to verify the following relation:

$$\left\langle T_{zz}(z)T_{ww}(w)\right\rangle = \frac{c/2}{(z-w)^4}. \tag{1}$$

Then:

$$\begin{aligned}
\left\langle T_{zz}(z)T_{ww}(w)\right\rangle &= \left\langle 0\right| \sum_n L_n\, z^{-n-2} \sum_m L_m\, w^{-m-2} \left|0\right\rangle \\
&= \left\langle 0\right| \sum_{n=2}^{\infty} L_n\, z^{-n-2} \sum_{m=-2}^{-\infty} L_m\, w^{-m-2} \left|0\right\rangle \\
&= \left\langle 0\right| \sum_{n=2}^{\infty} \sum_{m=-2}^{-\infty} z^{-n-2}\, w^{-m-2}\, L_n\, L_m \left|0\right\rangle \\
&= \left\langle 0\right| \sum_{n=2}^{\infty} \sum_{m=-2}^{-\infty} z^{-n-2}\, w^{-m-2}\, [L_n, L_m] \left|0\right\rangle \\
&= \left\langle 0\right| \sum_{n=2}^{\infty} \sum_{m=-2}^{-\infty} z^{-n-2} w^{-m-2} \left\{(n-m)L_{n+m} + \frac{c}{12}(n^3-n)\delta_{n+m,0}\right\} \left|0\right\rangle \\
&= \frac{c}{12}\left\langle 0\right| \sum_{n=2}^{\infty} \sum_{m=-2}^{-\infty} z^{-n-2} w^{-m-2} (n^3-n)\delta_{n+m,0} \left|0\right\rangle \\
&= \frac{c}{12}\left\langle 0\right| \sum_{n=2}^{\infty} z^{-n-2} w^{n-2} (n^3-n) \left|0\right\rangle \\
&= \frac{c}{12} \sum_{n=2}^{\infty} z^{-n-2} w^{n-2} (n^3-n) = \frac{c}{12}\frac{6}{(z-w)^4} \\
&= \frac{c/2}{(z-w)^4}.
\end{aligned} \tag{2}$$

We used the following:

- In the first line we used that

$$T_{zz}(z) = \sum_n L_n\, z^{-n-2}, \qquad T_{ww}(w) = \sum_m L_m\, w^{-m-2}. \tag{3}$$

- The second line is one of the most crucial steps. We use that :

$$L_m|0\rangle = 0, \quad m \geq -1 \tag{4}$$
$$\langle 0|L_n = 0, \quad n \leq 1. \tag{5}$$

  This forces the right sum to be from $-\infty$ to -2, otherwise it is automatically zero, and the left sum to be from 2 to ∞, otherwise it will be automatically zero.

- In the third line we moved the coordinates to the left and the operators to the right, being careful not to change the order of the operators.
- In the forth line we used the following:

$$[L_n, L_m] = L_n L_m - L_m L_n. \tag{6}$$

But using the same logic as the second line, $L_n|0\rangle = 0$ for $n \geq 2$ and the same for L_m to the left. So using our conditions we can formally say that in this case:

$$[L_n, L_m] = L_n L_m. \tag{7}$$

- In the fifth line we used that

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}. \tag{8}$$

- In the sixth line we used once more that any combination of m, n in L_{n+m} with $n \neq m$ will give zero, in other words:

$$\Big\langle 0\Big| \sum_{n=2}^{\infty} \sum_{m=-2}^{-\infty} (n-m) L_{n+m} \Big|0\Big\rangle = 0. \tag{9}$$

Feel free to try this.

- In the seventh line we used that

$$\delta_{n+m,0} = \begin{cases} 1, & n = -m \\ 0, & n \neq -m. \end{cases} \tag{10}$$

Hence, $n = -m$ and

$$\sum_{m=-2}^{-\infty} w^{-m-2} = \sum_{(-n)=-2}^{-\infty} w^{-(-n)-2} = \sum_{n=2}^{\infty} w^{n-2} \tag{11}$$

- In the last line we used the value of the sum, in other words:

$$\sum_{n=2}^{\infty} z^{-n-2} w^{n-2} (n^3 - n) = \frac{6}{(z-w)^4}. \tag{12}$$

For those who want to calculate the sum analytically and without a computational program:

- We start with $\sum\limits_{n=2}^{\infty} z^{-n-2}w^{n-2}(n^3-n)$ and we replace n with $n+2$.

$$\begin{aligned}\sum_{n=2}^{\infty} z^{-n-2}w^{n-2}(n^3-n) &= \sum_{n+2=2}^{\infty} z^{-(n+2)-2}w^{(n+2)-2}((n+2)^3-(n+2)) \\ &= \sum_{n=0}^{\infty} z^{-n-4}w^{n}((n+2)^3-(n+2)) \\ &= \sum_{n=0}^{\infty} z^{-n-4}w^{n}(n+2)\Big((n+2)^2-1\Big) \\ &= z^{-4}\sum_{n=0}^{\infty}(n+2)\Big((n+2)^2-1\Big)\left(\frac{w}{z}\right)^n \\ &= z^{-4}\sum_{n=0}^{\infty}(n+2)(n+3)(n+1)\left(\frac{w}{z}\right)^n.\end{aligned} \tag{13}$$

Now take the function:

$$f(x) = \sum_{n=0}^{\infty} x^{n+3} = x^3\sum_{n=0}^{\infty} x^n = \frac{x^3}{1-x}, \tag{14}$$

where we used that $\sum\limits_{n=0}^{\infty} x^n = \frac{1}{1-x}$. The desired results comes from $f'''(x)$. It is easy to see, if we take both sides.
Starting with the l.h.s, i.e. $f(x) = \sum\limits_{n=0}^{\infty} x^{n+3}$, we have:

1. $f'(x) = \sum\limits_{n=0}^{\infty}(n+3)x^{n+2}$,
2. $f''(x) = \sum\limits_{n=0}^{\infty}(n+3)(n+2)x^{n+1}$,
3. $f'''(x) = \sum\limits_{n=0}^{\infty}(n+3)(n+2)(n+1)x^{n}$.

Now from the r.h.s, i.e. $f(x) = \frac{x^3}{1-x}$, we find:

1. $f'(x) = \frac{-x^2(2x-3)}{(x-1)^2}$,
2. $f''(x) = \frac{-2x(x^2-3x+3)}{(x-1)^3}$,
3. $f'''(x) = \frac{6}{(x-1)^4}$.

Thus,

$$\sum_{n=0}^{\infty}(n+2)(n+3)(n+1)x^n = \frac{6}{(x-1)^4}. \tag{15}$$

- Hence, for $x = \frac{w}{z}$ we have:

$$\begin{aligned} z^{-4}\frac{6}{(x-1)^4} &= z^{-4}\frac{6}{(1-x)^4} = z^{-4}\frac{6}{(1-\frac{w}{z})^4} \\ &= z^{-4}\frac{6}{(\frac{z-w}{z})^4} \\ &= \frac{6}{(z-w)^4}. \end{aligned} \tag{16}$$

II. Highest weight states and the mode expansion

1. *Verify Eq. (3.89) in the notes.*

We want to verify that:

$$\begin{aligned} [L_n, \Phi(w,\bar{w})] &= \oint \frac{\mathrm{d}z}{2\pi i} z^{n+1} \mathcal{R}\Big(T_{zz}\Phi(w,\bar{w})\Big) \\ &= (n+1)hw^n\Phi(w,\bar{w}) + w^{n+1}\partial_w\Phi(w,\bar{w}). \end{aligned} \tag{17}$$

We shall use that $\mathcal{R}\Big(T_{zz}\Phi(w,\bar{w})\Big) = \frac{h}{(z-w)^2}\Phi(w,\bar{w}) + \frac{1}{z-w}\partial_w\Phi(w,\bar{w})$. Then, we have:

$$\begin{aligned} &\oint \frac{\mathrm{d}z}{2\pi i} z^{n+1}\mathcal{R}\Big(T_{zz}\Phi(w,\bar{w})\Big) = \oint \frac{\mathrm{d}z}{2\pi i} z^{n+1}\left(\frac{h}{(z-w)^2}\Phi(w,\bar{w}) + \frac{1}{z-w}\partial_w\Phi(w,\bar{w})\right) \\ &= \frac{1}{2\pi i}\oint \mathrm{d}z z^{n+1}\frac{h}{(z-W)^2}\Phi(w,\bar{w}) + \frac{1}{2\pi i}\oint \mathrm{d}z z^{n+1}\frac{1}{z-w}\partial_w\Phi(w,\bar{w}) \\ &= \frac{h}{2\pi i}\Phi(w,\bar{w})\oint \mathrm{d}z\frac{z^{n+1}}{(z-w)^2} + \frac{\partial_w\Phi(w,\bar{w})}{2\pi i}\oint \mathrm{d}z\frac{z^{n+1}}{z-w}. \end{aligned} \tag{18}$$

Everything comes down to calculating the following two integrals.

(a) The first contour integral that we have to evaluate is $\oint dz\frac{z^{n+1}}{(z-w)^2}$. This has a pole at $z=0$ of rank $m=n+1$, and a pole at $z=w$ of rank $m=2$. But the evaluation of the residue at $z=0$ gives zero. So we are only interested in the pole $z=w$ of rank $m=2$. Then, evaluating the residue, we have:

$$\begin{aligned} c &= \lim_{z\to w}\left\{\frac{\mathrm{d}}{\mathrm{d}z}\left((z-w)^2\frac{z^{n+1}}{(z-w)^2}\right)\right\} \\ &= \lim_{z\to w}\left[(n+1)z^n\right] = (n+1)w^n. \end{aligned} \tag{19}$$

Then, using Cauchy's principle, we can evaluate the contour integral as

$$\begin{aligned} \oint \mathrm{d}z\frac{z^{n+1}}{(z-w)^2} &= 2\pi i \times \left\{\text{Sum of residues}\right\} \\ &= 2\pi i \times c = 2\pi i(n+1)w^n. \end{aligned} \tag{20}$$

(b) The second integral that we have to calculate is $\oint dz \frac{z^{n+1}}{z-w}$. This has a single pole at $z = 2$ of rank $m = 1$ and a pole at $z = 0$ of rank $m = n + 1$. As before, the second pole has a zero residue, so only the first pole contributes to the integral. Thus we have to evaluate the residue at the pole $z = w$. We have:

$$c = \lim_{z \to w} \left\{ (z - w) \frac{z^{n+1}}{(z - w)} \right\} = w^{n+1}. \tag{21}$$

Then the integral can be evaluated as:

$$\oint \mathrm{d}z \frac{z^{n+1}}{z - w} = 2\pi i w^{n+1}. \tag{22}$$

Thus, plugging equation (20) and equation (22) into equation (18) we get that:

$$\begin{aligned} \oint \frac{\mathrm{d}z}{2\pi i} z^{n+1} \mathcal{R}\Big(T_{zz}\Phi(w, \bar{w})\Big) &= \frac{h}{2\pi i}\Phi(w, \bar{w}) 2\pi i (n + 1) w^n + \frac{\partial_w \Phi(w, \bar{w})}{2\pi i} 2\pi i w^{n+1} \\ &= (n + 1) h w^n \Phi(w, \bar{w}) + w^{n+1} \partial_w \Phi(w, \bar{w}). \end{aligned} \tag{23}$$

2. *Verify Eq. (3.93) in the notes.*

We have to verify that:

$$\langle h | L^\dagger_{-n} L_{-n} | h \rangle = \Big(2nh + \frac{c}{12}(n^3 - n) \Big) \langle h | h \rangle. \tag{24}$$

Our starting point is $\langle h | L^\dagger_{-n} L_{-n} | h \rangle$. Then we have:

$$\begin{aligned} \langle h | L^\dagger_{-n} \, L_{-n} | h \rangle &= \langle h | L_n \, L_{-n} | h \rangle \\ &= \langle h | \big[L_n, L_{-n} \big] | h \rangle \\ &= \langle h | 2nL_0 + \frac{c}{12}(n^3 - n) | h \rangle \\ &= 2n \langle h | L_0 | h \rangle + \frac{c}{12}(n^3 - n) \langle h | h \rangle \\ &= \Big(2nh + \frac{c}{12}(n^3 - n) \Big) \langle h | h \rangle. \end{aligned} \tag{25}$$

- In the first line we used

$$L^\dagger_m = L_{-m} \Rightarrow L_m = L^\dagger_{-m}. \tag{26}$$

- In the second line we use an argument similar to the first exercise. We know the following:

$$\big[L_n, \, L_{-n} \big] = L_n L_{-n} - L_{-n} L_n, \tag{27}$$

$$\langle h | L_n = 0, \quad n < 0, \tag{28}$$

$$L_n | h \rangle = 0, \quad n > 0 \Rightarrow L_{-n} | h \rangle = 0, \quad n < 0. \tag{29}$$

It is clear that in order to have a non-zero result we must have $n \geq 0$. But then equation (27) indicates that for $n \geq 0$ the second part of the r.h.s is zero, in other words:

$$\left[L_n,\, L_{-n}\right] = L_n L_{-n}. \tag{30}$$

- In the third line we use the definition of the commutator of the Virasoro algebra:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}, \tag{31}$$

  but for $m = -n$.
- In the fourth line we just separated the two parts.
- In the fifth line we used that:

$$L_0|h\rangle = h|h\rangle. \tag{32}$$