Series 11

I. Two derivations for the n-pt fn with one descendent field

1. *Verify Eq. (3.111) and (3.112) in the notes.*

We start from the conformal Ward identity:

$$\langle T(z)\phi_1(w_1)\dots\phi_n(w_n)\rangle = \sum_{j=1}^{n}\left[\frac{h_j}{(z-w_j)^2}+\frac{1}{z-w_j}\frac{\partial}{\partial w_j}\right]\langle\phi_1(w_1)\dots\phi_n(w_n)\rangle. \tag{1}$$

If we take the limit $z \to w_n$ we can use the definition

$$T(z)\phi_n(w_n) = \sum_{k\geq 0}(z-w_n)^{k-2}\hat{L}_{-k}\phi_n(w_n). \tag{2}$$

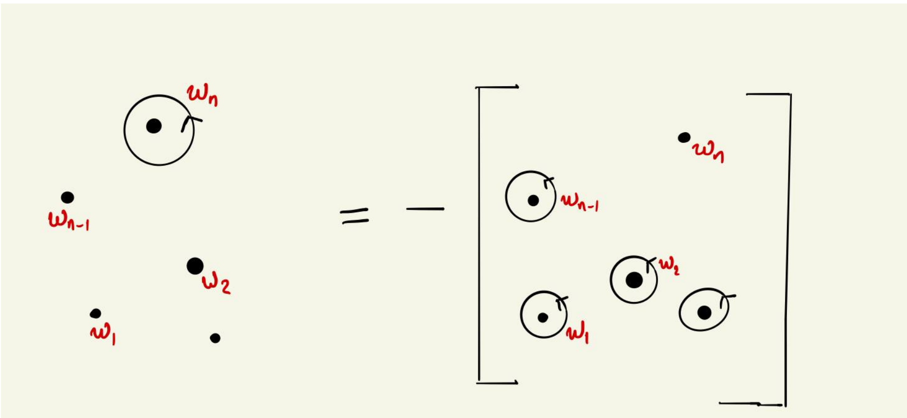
So using equation (2) the conformal ward identity takes the form:

$$\sum_{k\geq 0}(z-w_n)^{k-2}\langle\phi_1(w_1)\dots\hat{L}_{-k}\phi_n(w_n)\rangle = \sum_{j=1}^{n}\left[\frac{h_j}{(z-w_j)^2}+\frac{1}{z-w_j}\frac{\partial}{\partial w_j}\right]\langle\phi_1(w_1)\dots\phi_n(w_n)\rangle. \tag{3}$$

Now, we can use the Cauchy formula and get:

$$\langle\phi_1(w_1)\dots\hat{L}_{-k}\phi_n(w_n)\rangle = \oint\frac{\mathrm{d}z}{2\pi i}(z-w_n)^{1-k}\times \left[\sum_{j=1}^{n}\left[\frac{h_j}{(z-w_j)^2}+\frac{1}{z-w_j}\frac{\partial}{\partial w_j}\right]\langle\phi_1(w_1)\dots\phi_n(w_n)\rangle\right]. \tag{4}$$

Now, since the residue at infinity vanishes, we can use the residue theorem to express the contour around the point w_n in terms of the contour integrals around the points w_i, $i = 1,\dots,n-1$ in clockwise fashion. Of course, we can instead take it as minus times the usual anticlockwise fashion, i.e. put a $-$ in front of the expression. (See figure.)



Thus:

$$\left\langle \phi_1(w_1)\dots \hat{L}_{-k}\phi_n(w_n)\right\rangle = -\oint\limits_{w_j} \frac{dz}{2\pi i}(z-w_n)^{1-k}\times \left[\sum_{j=1}^{n-1}\left[\frac{h_j}{\left(z-w_j\right)^2}+\frac{1}{z-w_j}\frac{\partial}{\partial w_j}\right]\left\langle \phi_1(w_1)\dots\phi_n(w_n)\right\rangle\right]. \tag{5}$$

Now we should calculate the following integrals:

$$\sum_{j=1}^{n-1}\oint\limits_{w_j}\frac{dz}{2\pi i}(z-w_n)^{1-k}\frac{1}{\left(z-w_j\right)^2}, \tag{6}$$

$$\sum_{j=1}^{n-1}\oint\limits_{w_j}\frac{dz}{2\pi i}(z-w_n)^{1-k}\frac{1}{z-w_j}. \tag{7}$$

We should use the following formula:

$$\oint\limits_{w}\frac{dz}{2\pi i}\frac{1}{(z-w)^n}f(z)=\frac{1}{(n-1)!}\partial^{n-1}f(w). \tag{8}$$

Hence,

$$\begin{aligned}\sum_{j=1}^{n-1}\oint\limits_{w_j}\frac{dz}{2\pi i}\frac{1}{(z-w_n)^{k-1}}\frac{1}{(z-w_j)^2} &= \frac{1}{1!}\frac{\partial}{\partial z}(z-w_n)^{1-k}|_{z=w_j}\\ &= \sum_{j=1}^{n-1}-\frac{k-1}{(w_j-w_n)^{k-2}}\\ &= \sum_{j=1}^{n-1}\frac{(1-k)}{(w_j-w_n)^{k-2}}.\end{aligned} \tag{9}$$

The other integral is:

$$\begin{aligned}\sum_{j=1}^{n-1}\oint\limits_{w_j}\frac{dz}{2\pi i}\frac{1}{(z-w_n)^{k-1}}\frac{1}{(z-w_j)} &= \sum_{j=1}^{n-1}(z-w_n)^{1-k}|_{z=w_j}\\ &= \sum_{j=1}^{n-1}\frac{1}{(w_j-w_n)^{k-1}}.\end{aligned} \tag{10}$$

So finally plugging equation (9) and equation (10) back into equation (5) we get:

$$\begin{aligned}\left\langle \phi_1(w_1)\dots \hat{L}_{-k}\phi_n(w_n)\right\rangle = -\sum_{j=1}^{n-1}\left\{\frac{(1-k)h_j}{(w_j-w_n)^{k-2}}+\frac{1}{(w_j-w_n)^{k-1}}\frac{\partial}{\partial w_j}\right\}\times\\ \times\left\langle \phi_1(w_1)\dots\phi_n(w_n)\right\rangle.\end{aligned} \tag{11}$$

In other words, putting back $w_n \to z$ we have:

$$\left\langle \phi_1(w_1)\ldots\phi_{n-1}(w_{n-1})\hat{L}_{-k}\phi(z)\right\rangle = \mathcal{L}_{-k}\left\langle \phi_1(w_1)\ldots\phi_{n-1}(w_{n-1})\phi(z)\right\rangle, \tag{12}$$

where:

$$\mathcal{L}_{-k} = -\sum_{j=1}^{n-1}\left\{\frac{(1-k)h_j}{(w_j-z)^{k-2}} + \frac{1}{(w_j-z)^{k-1}}\frac{\partial}{\partial w_j}\right\}. \tag{13}$$

2. *Verify Eq. (3.111) and (3.112) at $z=0$ via the alternative derivation sketched above (3.113).*

We start with the expression

$$\left\langle 0\middle|\phi_1(w_1)\ldots\phi_{n-1}(w_{n-1})\hat{L}_{-k}\phi(z)\middle|0\right\rangle. \tag{14}$$

For $z=0$ we use

$$\hat{L}_{-k}\phi(0) = L_{-k}\phi(0). \tag{15}$$

In other words:

$$\left\langle 0\middle|\phi_1(w_1)\ldots\phi_{n-1}(w_{n-1})\hat{L}_{-k}\phi(0)\middle|0\right\rangle = \left\langle 0\middle|\phi_1(w_1)\ldots\phi_{n-1}(w_{n-1})L_{-k}\phi(0)\middle|0\right\rangle. \tag{16}$$

But we know that

$$[L_n,\phi(w_j)] = h_j(n+1){w_j}^n\phi(w_j) + {w_j}^{n+1}\partial w_j\phi(w_j). \tag{17}$$

So for $n=-k$ we have:

$$[L_{-k},\phi(w_j)] = h_j(-k+1){w_j}^{-k}\phi(w_j) + {w_j}^{-k+1}\partial w_j\phi(w_j). \tag{18}$$

Now we are going to use

$$[L_{-k},\phi(w_j)] = \Big(h_j(-k+1){w_j}^{-k} + {w_j}^{-k+1}\partial w_j\Big)\phi(w_j), \tag{19}$$

$$\phi(w_j)L_{-k} = L_{-k}\phi(w_j) - [L_{-k},\phi(w_j)]. \tag{20}$$

Thus for $\phi_{n-1}(w_{n-1})L_{-k}$ we have:

$$\phi_{n-1}(w_{n-1})L_{-k} = L_{-k}\phi_{n-1}(w_{n-1}) - \left[\frac{h_{n-1}(1-k)}{(w_{n-1})^k} + \frac{\partial w_{n-1}}{(w_{n-1})^{k-1}}\right]\phi_{n-1}(w_{n-1}) \tag{21}$$

so we can write equation (16) as:

$$\begin{aligned}\left\langle \phi_1(w_1)\ldots\phi_{n-1}(w_{n-1})L_{-k}\phi(0)\right\rangle = &\\ &\left\langle \phi_1(w_1)\ldots L_{-k}\phi_{n-1}(w_{n-1})\phi(0)\right\rangle \\ &- \left[\frac{h_{n-1}(1-k)}{(w_{n-1})^k} + \frac{\partial w_{n-1}}{(w_{n-1})^{k-1}}\right]\left\langle \phi_1(w_1)\ldots\phi_{n-1}(w_{n-1})\phi(0)\right\rangle.\end{aligned} \tag{22}$$

Keep commuting L_{-k} all the way to the left, we end up with:

$$\begin{aligned}\big\langle 0\big|\phi_1(w_1)\dots\phi_{n-1}(w_{n-1})L_{-k}\phi(0)\big|0\big\rangle = \\ \big\langle 0\big|L_{-k}\phi_1(w_1)\dots\phi_{n-1}(w_{n-1})\phi(0)\big|0\big\rangle \\ -\sum_{j=1}^{n-1}\left[\frac{h_j(1-k)}{(w_j)^k}+\frac{\partial w_j}{(w_j)^{k-1}}\right]\big\langle\phi_1(w_1)\dots\phi_{n-1}(w_{n-1})\phi(0)\big\rangle.\end{aligned} \tag{23}$$

Using that:

$$\big\langle 0\big|L_{-k}=0, \qquad k>2, \tag{24}$$

the r.h.s term in the first line of equation (23) vanishes and thus we end up with:

$$\begin{aligned}\big\langle 0\big|\phi_1(w_1)\dots\phi_{n-1}(w_{n-1})L_{-k}\phi(0)\big|0\big\rangle = -\sum_{j=1}^{n-1}\left[\frac{h_j(1-k)}{(w_j)^k}+\frac{\partial w_j}{(w_j)^{k-1}}\right]\times \\ \times\big\langle\phi_1(w_1)\dots\phi_{n-1}(w_{n-1})\phi(0)\big\rangle.\end{aligned} \tag{25}$$

Using equation (15) we conclude that:

$$\big\langle 0\big|\phi_1(w_1)\dots\phi_{n-1}(w_{n-1})\hat{L}_{-k}\phi(0)\big|0\big\rangle = \mathcal{L}_{-k}\big\langle 0\big|\phi_1(w_1)\dots\phi_{n-1}(w_{n-1})\phi(0)\big|0\big\rangle, \tag{26}$$

$$\mathcal{L}_{-k} = -\sum_{j=1}^{n-1}\left[\frac{h_j(1-k)}{(w_j)^k}+\frac{\partial w_j}{(w_j)^{k-1}}\right]. \tag{27}$$

II. The 2-pt function of descendants of different primary fields vanishes

Show this (see (3.114) and following)

Start with the definition of a descendant state that includes the Ward identity:

$$\begin{aligned}\big\langle\phi_1(w_1)\hat{L}_{-k}\phi_2(w_2)\big\rangle = \oint\frac{\mathrm{d}z}{2\pi i}(z-w_2)^{1-k}\times \\ \left[\sum_{j=1}^{2}\left[\frac{h_j}{\left(z-w_j\right)^2}+\frac{1}{z-w_j}\frac{\partial}{\partial w_j}\right]\big\langle\phi_1(w_1)\phi_2(w_2)\big\rangle\right].\end{aligned} \tag{28}$$

Then if we use

$$\big\langle\phi_1(w_1)\phi_2(w_2)\big\rangle = \frac{\delta_{12}}{(w_1-w_2)^{2h_1}}, \tag{29}$$

it is clear from the Ward identity that if we start we different primary fields, the two-point function of the descendant field will vanish. In other words the orthogonality condition of the primary fields automatically implies that also the two-point correlation functions of the descendant fields of two different families vanish. There is a complete orthogonality between two different conformal families.

III. $T(z)T(w)$ for the free massless boson

Calculate the OPE of T with itself for the free massless boson, see (4.13).

We will use the following equations:

$$T(z) = -2\pi g : \partial_z\phi\partial_z\phi :, \tag{30}$$

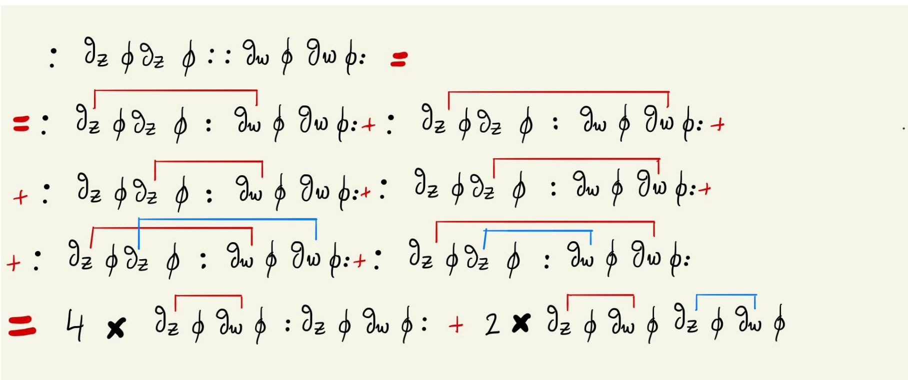
$$T(w) = -2\pi g : \partial_w\phi\partial_w\phi :, \tag{31}$$

$$\mathcal{R}\Big(\partial_z\phi\partial_w\phi\Big) = -\frac{1}{4\pi g}\frac{1}{(z-w)^2}. \tag{32}$$

Thus we have:

$$\begin{aligned}
\mathcal{R}\Big(T(z)T(w)\Big) &= 4\pi^2 g^2\Big(: \partial_z\phi\partial_z\phi :: \partial_w\phi\partial_w\phi : \Big) \\
&= 4\pi^2 g^2\Big(2 \times \frac{1}{16\pi^2 g^2}\frac{1}{(z-w)^4} - 4 \times \frac{1}{4\pi g}\frac{1}{(z-w)^2} : \partial_z\phi\partial_w\phi : \Big) \\
&= \frac{1/2}{(z-w)^4} - 4\pi g\frac{: \partial_z\phi\partial_w\phi :}{(z-w)^2} \\
&= \frac{1/2}{(z-w)^4} + \frac{2\Big(-2\pi g : \partial_z\phi\partial_w\phi :\Big)}{(z-w)^2} \\
&= \frac{1/2}{(z-w)^4} + \frac{2\Big(-2\pi g : \partial_w\phi\partial_w\phi :\Big)}{(z-w)^2} - \frac{4\pi g(z-w) : \partial_w\Big(\partial_w\phi\Big)\partial_w\phi}{(z-w)^2} \\
&= \frac{1/2}{(z-w)^4} + \frac{2\Big(-2\pi g : \partial_w\phi\partial_w\phi :\Big)}{(z-w)^2} - \frac{4\pi g : \partial_w^2\phi\partial_w\phi}{(z-w)} \\
&= \frac{1/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)}.
\end{aligned} \tag{33}$$

1. To go from the first line to the second we used the contractions as in the picture and then the definitions.



2. To go from the fourth line to the fifth we Taylor expanded the field as:

$$\partial_z\phi = \partial_w\phi + (z-w)\Big\{\partial_z\left(\partial_z\phi\right)\Big|_{z=w}\Big\}. \tag{34}$$

3. In the last line we used that:

$$\begin{aligned} \partial_w T(w) &= -2\pi g : \partial_w^2 \phi \partial_w \phi : -2\pi g : \partial_w \phi \partial_w^2 \phi : \\ &= -4\pi g : \partial_w^2 \phi \partial_w \phi : . \end{aligned} \tag{35}$$