Series 12

I. Creation an annihilation operators for the free boson on the cylinder

1. *Verify Eq. (4.21) for the Hamiltonian in the notes.*

   We want to show that:

$$\begin{aligned} H &= \frac{1}{2gL}\sum_n \left(\pi_n\pi_{-n} + (2\pi g n)^2\phi_n\phi_{-n}\right) \\ &= \frac{1}{2gL}\pi_0^2 + \frac{2\pi}{L}\sum_{n>0}\left(\alpha_{-n}\alpha_n + \bar{\alpha}_{-n}\bar{\alpha}_n\right). \end{aligned} \tag{1}$$

   We start from the l.h.s of equation (1) and split into positive and negative n and take the case for $n = 0$ separately. Thus we have:

$$\begin{aligned} H = \frac{1}{2gL}\left(\pi_n\pi_{-n} + (2\pi g n)^2\phi_n\phi_{-n}\right)|_{n=0} + \frac{1}{2gL}\Bigg\{ \sum_{n>0}\left(\pi_n\pi_{-n} + (2\pi g n)^2\phi_n\phi_{-n}\right) \\ + \sum_{n<0}\left(\pi_n\pi_{-n} + (2\pi g n)^2\phi_n\phi_{-n}\right)\Bigg\} \\ = \frac{1}{2gL}\pi_0^2 + \frac{1}{2gL}\Bigg\{ \sum_{n>0}\Big(\pi_n\pi_{-n} + (2\pi g n)^2\phi_n\phi_{-n} + \pi_{-n}\pi_n + (2\pi g n)^2\phi_{-n}\phi_n\Big)\Bigg\}. \end{aligned} \tag{2}$$

   Now we can clearly see that the first component of equation (2) is the same with the first component of the second line of equation (1). So now, we need to show that:

$$\begin{aligned} \frac{2\pi}{L}\sum_{n>0}\left(\alpha_{-n}\alpha_n + \bar{\alpha}_{-n}\bar{\alpha}_n\right) = \\ \frac{1}{2gL}\Bigg\{ \sum_{n>0}\Big(\pi_n\pi_{-n} + (2\pi g n)^2\left(\phi_n\phi_{-n} + \phi_{-n}\phi_n\right) + \pi_{-n}\pi_n\Big)\Bigg\}. \end{aligned} \tag{3}$$

We start from the l.h.s of equation (3) and use the definitions of α. In general we have:

$$\alpha_n = \begin{cases} -i\sqrt{n}\tilde{\alpha}_n, & n > 0 \\ i\sqrt{-n}\tilde{\alpha}^\dagger_{-n}, & n < 0 \end{cases} \qquad \bar{\alpha}_n = \begin{cases} -i\sqrt{n}\tilde{\alpha}_{-n}, & n > 0 \\ i\sqrt{-n}\tilde{\alpha}^\dagger_{n}, & n < 0 \end{cases} \tag{4}$$

$$\alpha_{-n} = \begin{cases} -i\sqrt{-n}\tilde{\alpha}_{-n}, & n < 0 \\ i\sqrt{n}\tilde{\alpha}^\dagger_{n}, & n > 0 \end{cases} \qquad \bar{\alpha}_{-n} = \begin{cases} -i\sqrt{-n}\tilde{\alpha}_{n}, & n < 0 \\ i\sqrt{n}\tilde{\alpha}^\dagger_{-n}, & n > 0 \end{cases} \tag{5}$$

$$\tilde{\alpha}_n = \frac{1}{\sqrt{4\pi g\,|n|}}\left(2\pi g\,|n|\,\phi_n + i\pi_{-n}\right) \tag{6}$$

$$\tilde{\alpha}^\dagger_n = \frac{1}{\sqrt{4\pi g\,|n|}}\left(2\pi g\,|n|\,\phi^\dagger_n - i\pi^\dagger_{-n}\right) \tag{7}$$

$$\phi^\dagger_n = \phi_{-n} \tag{8}$$

$$\phi_n = \phi^\dagger_{-n} \tag{9}$$

$$\pi^\dagger_n = \pi_{-n} \tag{10}$$

$$\pi_n = \pi^\dagger_{-n}. \tag{11}$$

So, for $n > 0$,

- For $\alpha_{-n}\alpha_n$ we have:

$$\begin{aligned} \alpha_{-n}\alpha_n &= \left(i\sqrt{n}\tilde{\alpha}^\dagger_n\right)\left(-i\sqrt{n}\tilde{\alpha}_n\right) \\ &= -i^2\left(\sqrt{n}\right)^2\tilde{\alpha}^\dagger_n\tilde{\alpha}_n \\ &= \frac{|n|}{4\pi g\,|n|}\left[\left(2\pi g\,|n|\,\phi_{-n} - i\pi_n\right)\left(2\pi g\,|n|\,\phi_n + i\pi_{-n}\right)\right] \\ &= \frac{1}{4\pi g}\left[\left(2\pi g\,|n|\,\right)^2\phi_{-n}\phi_n + \pi_n\pi_{-n} - 2i\pi g\,|n|\,\left(\pi_n\phi_n - \pi_{-n}\phi_{-n}\right)\right]. \end{aligned} \tag{12}$$

- For $\bar{\alpha}_{-n}\bar{\alpha}_n$ we have:

$$\begin{aligned} \bar{\alpha}_{-n}\bar{\alpha}_n &= \left(i\sqrt{n}\tilde{\alpha}^\dagger_{-n}\right)\left(-i\sqrt{n}\tilde{\alpha}_{-n}\right) \\ &= -i^2\left(\sqrt{n}\right)^2\tilde{\alpha}^\dagger_{-n}\tilde{\alpha}_{-n} \\ &= \frac{|n|}{4\pi g\,|n|}\left[\left(2\pi g\,|n|\,\phi_{n} - i\pi_{-n}\right)\left(2\pi g\,|n|\,\phi_{-n} + i\pi_{n}\right)\right] \\ &= \frac{1}{4\pi g}\left[\left(2\pi g\,|n|\,\right)^2\phi_{n}\phi_{-n} + \pi_{-n}\pi_{n} + 2i\pi g\,|n|\,\left(\pi_n\phi_n - \pi_{-n}\phi_{-n}\right)\right]. \end{aligned} \tag{13}$$

So if we use the l.h.s of equation (3) and use equation (12) and equation (13) we get:

$$\begin{aligned}
&\frac{2\pi}{L}\sum_{n>0}\left(\alpha_{-n}\alpha_n+\bar{\alpha}_{-n}\bar{\alpha}_n\right)= \\
&=\frac{2\pi}{L}\frac{1}{4\pi g}\sum_{n>0}\Bigg(\left(2\pi g\,|n|\right)^2\left(\phi_{-n}\phi_n+\phi_n\phi_{-n}\right)+\pi_n\pi_{-n}+\pi_{-n}\pi_n \\
&-2i\pi g\,|n|\left(\pi_n\phi_n-\pi_{-n}\phi_{-n}\right)+2i\pi g\,|n|\left(\pi_n\phi_n-\pi_{-n}\phi_{-n}\right)\Bigg) \\
&=\frac{1}{2gL}\sum_{n>0}\left(\left(2\pi g\,|n|\right)^2\left(\phi_{-n}\phi_n+\phi_n\phi_{-n}\right)+\pi_n\pi_{-n}+\pi_{-n}\pi_n\right),
\end{aligned} \tag{14}$$

which verifies the desired result.

2. *Verify the commutation relation Eq. (4.22) in the notes.*

We want to show that

$$\left[H,\alpha_{-m}\right]=\frac{2\pi}{L}m\alpha_{-m}. \tag{15}$$

Start with the commutator and use that

$$H=\frac{1}{2gL}\pi_0^2+\frac{2\pi}{L}\sum_{n>0}\left(\alpha_{-n}\alpha_n+\bar{\alpha}_{-n}\bar{\alpha}_n\right). \tag{16}$$

We have the following commutation rules:

$$\left[\pi_0,\alpha_n\right]=0, \tag{17}$$
$$\left[\alpha_n,\bar{\alpha}_m\right]=0, \tag{18}$$
$$\left[\alpha_n,\alpha_m\right]=n\delta_{n+m,0}, \tag{19}$$
$$\left[\bar{\alpha}_n,\bar{\alpha}_m\right]=n\delta_{n+m,0}. \tag{20}$$

Thus, the l.h.s of equation (15) becomes

$$\left[H,\alpha_{-m}\right]=\frac{2\pi}{L}\sum_{n>0}\left[\left(\alpha_{-n}\alpha_n+\bar{\alpha}_{-n}\bar{\alpha}_n\right),\alpha_{-m}\right]. \tag{21}$$

We have to compute the following:

(a) First:

$$\begin{aligned}
\left[(\alpha_{-n}\,\alpha_n),\alpha_{-m}\right]&=\alpha_{-n}\left[\alpha_n,\,\alpha_{-m}\right]+\left[\alpha_{-n},\,\alpha_{-m}\right]\alpha_n \\
&=\alpha_{-n}\,n\,\delta_{n-m,0}\,-n\,\delta_{-n-m,0}\,\alpha_n.
\end{aligned} \tag{22}$$

(b) Second:

$$\left[\bar{\alpha}_{-n}\bar{\alpha}_n,\alpha_{-m}\right]=0. \tag{23}$$

So, equation (22) has two possible outcomes:

(a) For $n = m$ we see that $\delta_{n-m,0} = 1$ and $\delta_{-n-m,0} = 0$, thus:

$$\left[\alpha_{-n}\alpha_n, \alpha_{-m}\right] = m\alpha_{-m}. \tag{24}$$

(b) For $n = -m$ we see that $\delta_{n-m,0} = 0$ and $\delta_{-n-m,0} = 1$, thus:

$$\left[\alpha_{-n}\alpha_n, \alpha_{-m}\right] = m\alpha_{-m}. \tag{25}$$

But the sum in equation (21) is only for $n > 0$ thus this solution is invalid and we can omit it.

(c) Hence, for any other n besides $n = m$ we have that:

$$\left[(\alpha_{-n}\,\alpha_n), \alpha_{-m}\right] = 0. \tag{26}$$

So we can omit the sum over n in equation (21) and we end up with:

$$\left[H, \alpha_{-m}\right] = \frac{2\pi}{L} m\alpha_{-m}, \quad m > 0. \tag{27}$$

3. *Verify the form of the Fourier modes Eq. (4.23) in terms of a_n, $\bar{a}_{-n}$.*

We want to show that:

$$\phi_n = \frac{i}{n\sqrt{4\pi g}}\left(\alpha_n - \bar{\alpha}_{-n}\right). \tag{28}$$

We start with the r.h.s of equation (42) and we take the cases for $n > 0$ and $n < 0$ separately.

- For $n > 0$ we have:

$$\begin{aligned}
\frac{i}{n\sqrt{4\pi g}}\left(\alpha_n - \bar{\alpha}_{-n}\right) &= \frac{i}{n\sqrt{4\pi g}}\left(-i\sqrt{n}\tilde{\alpha}_n - i\sqrt{n}\tilde{\alpha}^\dagger_{-n}\right) \\
&= \frac{\sqrt{n}\tilde{\alpha}_n}{n\sqrt{4\pi g}} + \frac{\sqrt{n}\tilde{\alpha}^\dagger_{-n}}{n\sqrt{4\pi g}} \\
&= \frac{1}{\sqrt{4\pi g n}}\left(\tilde{\alpha}_n + \tilde{\alpha}^\dagger_{-n}\right) \\
&= \frac{1}{\sqrt{4\pi g n}}\left(\frac{1}{\sqrt{4\pi g n}}\left(2\pi g n\phi_n + 2\pi g n\phi_n + i\pi_{-n} - i\pi_{-n}\right)\right) \\
&= \phi_n.
\end{aligned} \tag{29}$$

- For $n < 0$ we have:

$$\begin{aligned}
\frac{i}{n\sqrt{4\pi g}}\left(\alpha_n - \bar{\alpha}_{-n}\right) &= \frac{i}{n\sqrt{4\pi g}}\left(i\sqrt{-n}\tilde{\alpha}^\dagger_{-n} - \left(-i\sqrt{-n}\tilde{\alpha}_n\right)\right)\\
&= \frac{i}{n\sqrt{4\pi g}}\left(i\sqrt{-n}\tilde{\alpha}^\dagger_{-n} + i\sqrt{-n}\tilde{\alpha}_n\right)\\
&= \frac{i}{-m\sqrt{4\pi g}}\left(i\sqrt{m}\tilde{\alpha}^\dagger_{m} + i\sqrt{m}\tilde{\alpha}_{-m}\right)\\
&= -\frac{i^2\sqrt{m}}{m\sqrt{4\pi g}}\left(\tilde{\alpha}^\dagger_{m} + \tilde{\alpha}_{-m}\right)\\
&= \frac{1}{\sqrt{4\pi g m}}\left(\tilde{\alpha}_{-m} + \tilde{\alpha}^\dagger_{m}\right)\\
&= \frac{1}{\sqrt{4\pi g m}}\left(\frac{1}{\sqrt{4\pi g m}}\left(2\pi g m\phi_{-m} + 2\pi g m\phi_{-m} + i\pi_m - i\pi_m\right)\right)\\
&= \phi_{-m}\\
&= \phi_n,
\end{aligned} \tag{30}$$

where to go from the second line to the third we used that for $n > 0$ we set $n = -m,\ m > 0$. We have shown the desired result.

II. OPEs with vertex operators

1. *Verify the OPE $\partial\phi(z)\mathcal{V}_\alpha(w,\bar{w})$ Eq. (4.35).*

We want to show that :

$$\partial\phi(z)\mathcal{V}_\alpha(w,\bar{w}) = \frac{-i\alpha}{4\pi g}\frac{\mathcal{V}_\alpha(w,\bar{w})}{z-w}. \tag{31}$$

We know that:

$$\mathcal{V}_\alpha(z,\bar{z}) =: e^{i\alpha\phi(z,\bar{z})} : \tag{32}$$

So we can use the expression $\exp(ax) = \sum_{n=0}^{\infty}\frac{a^n x^n}{n!}$ and find

$$\exp\left(i\alpha\phi(w,\bar{w})\right) = \sum_{n=0}^{\infty}\frac{(i\alpha)^n\left(\phi(w,\bar{w})\right)^n}{n!}. \tag{33}$$

Thus we have:

$$
\begin{aligned}
\partial\phi(z)\mathcal{V}_\alpha(w,\bar{w}) &= \partial\phi(z) : \exp\left(ia\phi(w,\bar{w})\right) : \\
&= \partial\phi(z) : \sum_{n=0}^{\infty} \frac{(ia)^n \left(\phi(w,\bar{w})\right)^n}{n!} : \\
&= \sum_{n=0}^{\infty} \frac{(ia)^n}{n!} \Big[\partial\phi(z) : \phi(w,\bar{w}))^n : \Big] \\
&= \frac{(ia)^0}{0!}\Big\{\partial\phi(z) : \left(\phi(w,\bar{w})\right)^0 : \Big\} + \sum_{n=1}^{\infty} \frac{(ia)^n}{n!} \Big\{\partial\phi(z) : \left(\phi(w,\bar{w})\right)^n : \Big\} \\
&= \overbracket{\partial\phi(z) : 1} : + \sum_{n=1}^{\infty} \frac{(ia)^n}{n!} \Big\{\overbracket{\partial\phi(z) : \underbrace{\phi(w,\bar{w}), \ldots}}_{n \text{ times}}, \phi(w,\bar{w}) : \Big\} \\
&= \langle\phi(z)\rangle + \sum_{n=1}^{\infty} \frac{(ia)^n}{n!} \Big\{\overbracket{\partial\phi(z) : \phi}\underbrace{(w,\bar{w})\phi(w,\bar{w}), \ldots, \phi(w,\bar{w})}_{n \text{ times}} : + \\
&+ \overbracket{\partial\phi(z) : \phi(w,\bar{w})\phi}\underbrace{(w,\bar{w}), \ldots, \phi(w,\bar{w})}_{n \text{ times}} : \\
&+ \overbracket{\partial\phi(z) : \phi(w,\bar{w})\phi(w,\bar{w}), \ldots, \phi}\underbrace{(w,\bar{w})}_{n \text{ times}} : \Big\} \\
&= \sum_{n=1}^{\infty} \frac{(ia)^n}{n!} \Big\{ n \times \left(\overbracket{\partial\phi(z)\phi}(w,\bar{w})\right) : \left(\phi(w,\bar{w})\right)^{n-1} : \Big\} \\
&= -\frac{1}{4\pi g}\frac{1}{z-w} \sum_{n=1}^{\infty} \left\{ \frac{n(ia)^n}{n!} : \left(\phi(w,\bar{w})\right)^{n-1} : \right\} \\
&= -\frac{ia}{4\pi g}\frac{1}{z-w} \sum_{n=1}^{\infty} \left\{ \frac{(ia)^{n-1}}{(n-1)!} : \left(\phi(w,\bar{w})\right)^{n-1} : \right\} \\
&= -\frac{ia}{4\pi g}\frac{1}{z-w} \sum_{m+1=1}^{\infty} \left\{ \frac{(ia)^{(m+1)-1}}{((m+1)-1)!} : \phi(w,\bar{w})\right)^{(m+1)-1} : \right\} \\
&= -\frac{ia}{4\pi g}\frac{1}{z-w} : \exp\left(ia\phi(w,\bar{w})\right) : \\
&= -\frac{ia}{4\pi g}\frac{\mathcal{V}_\alpha(w,\bar{w})}{z-w}.
\end{aligned}
\tag{34}
$$

Where we used:

$$\langle \phi(z) \rangle = 0 \tag{35}$$

$$\langle \phi(z)\phi(w) \rangle = -\frac{1}{4\pi g}\ln(z-w) \tag{36}$$

$$\langle \partial\phi(z)\phi(w) \rangle = -\frac{1}{4\pi g}\frac{1}{z-w} \tag{37}$$

$$n! = 1\times 2\times\cdots\times n-1\times n \tag{38}$$

$$\frac{n}{n!} = \frac{n}{1\times\cdots\times n-1\times n} = \frac{1}{1\times\cdots\times n-1} = \frac{1}{(n-1)!} \tag{39}$$

2. *Verify the OPE $T(z)\mathcal{V}_\alpha(w,\bar{w})$ Eq. (4.36).*

We want to show that

$$T(z)\mathcal{V}_\alpha(w,\bar{w}) = \frac{a^2}{8\pi g}\frac{\mathcal{V}_\alpha(w,\bar{w})}{(z-w)^2} + \frac{\partial_w\mathcal{V}_\alpha(w,\bar{w})}{z-w}, \tag{40}$$

where

$$T(z) = -2\pi g : \partial\phi(z)\partial\phi(z) : . \tag{41}$$

Thus we have:

$$\begin{aligned}
T(z)\mathcal{V}_\alpha(w,\bar{w}) &= -2\pi g : \partial\phi(z)\partial\phi(z) : \mathcal{V}_\alpha(w,\bar{w}) \\
&= -2\pi g : \partial\phi(z)\partial\phi(z) :: \exp\big(ia\phi(w,\bar{w})\big) : \\
&= -2\pi g : \partial\phi(z)\partial\phi(z) :: \sum_{n=0}^{\infty}\frac{(ia)^n\big(\phi(w,\bar{w})\big)^n}{n!} : \\
&= -2\pi g \sum_{n=0}^{\infty}\frac{(ia)^n}{n!}\Big\{ : \partial\phi(z)\partial\phi(z) :: \big(\phi(w,\bar{w})\big)^n : \Big\} \\
&= -2\pi g : \partial\phi(z)\partial\phi(z) :: \big(\phi(w,\bar{w})\big)^0 : \\
&\quad - 2\pi g \sum_{n=1}^{\infty}\frac{(ia)^n}{n!}\Big\{ : \partial\phi(z)\partial\phi(z) :: \phi(w,\bar{w})\big(\phi(w,\bar{w})\big)^{n-1} : \Big\} \\
&\quad - 2\pi g \sum_{n=2}^{\infty}\frac{(ia)^n}{n!}\Big\{ : \partial\phi(z)\partial\phi(z) :: \phi(w,\bar{w})\phi(w,\bar{w})\big(\phi(w,\bar{w})\big)^{n-2} : \Big\} \\
&= -2\pi g \sum_{n=1}^{\infty}\frac{(ia)^n}{n!}\Big\{ : \overline{\partial\phi(z)\partial\phi(z) :: \phi(w,\bar{w}) \times \ldots}\ldots \times \big(\phi(w,\bar{w})\big)^{n-1} : \Big\} \\
&\quad - 2\pi g \sum_{n=1}^{\infty}\frac{(ia)^n}{n!}\Big\{ : \partial\phi(z)\overline{\partial\phi(z) :: \phi(w,\bar{w}) \times \ldots}\ldots \times \big(\phi(w,\bar{w})\big)^{n-1} : \Big\} \\
&\quad - 2\pi g \sum_{n=2}^{\infty}\frac{(ia)^n}{n!}\Big\{ : \overline{\partial\phi(z)\partial\phi(z) :: \phi(w,\bar{w})\phi(w,\bar{w})}\big(\phi(w,\bar{w})\big)^{n-2} : \Big\} \\
&= -2\pi g \sum_{n=1}^{\infty}\frac{(ia)^n}{n!}\Big\{ 2n\times\Big(\overline{\partial\phi(z)\phi}(w,\bar{w})\Big) : \partial\phi(z)\Big(\phi(w,\bar{w})\Big)^{n-1} : \Big\} \\
&\quad - 2\pi g \sum_{n=2}^{\infty}\frac{(ia)^n}{n!}\Big\{ n(n-1)\times\Big(\overline{\partial\phi(z)\phi}(w,\bar{w})\overline{\partial\phi(z)\phi}(w,\bar{w})\Big) : \Big(\phi(w,\bar{w})\Big)^{n-2} : \Big\}
\end{aligned} \tag{42}$$

If you are unsure about the coefficients in front you can convince yourselves by taking the simplest case, $n = 3$. Then you can actually count how many single and how many double contractions there are. Also in the double contraction, the sum has to start from $n = 2$ since we need at least two fields to be able to have a double contraction. Then we can use $\langle \partial\phi(z)\phi(w)\rangle = -\frac{1}{4\pi g}\frac{1}{z-w}$in which case equation (42) becomes:

$$\begin{aligned}
T(z)\mathcal{V}_\alpha(w,\bar{w}) &= -\frac{1}{8\pi g(z-w)^2}\sum_{n=2}^{\infty}\left\{\frac{(ia)^n}{n!}n(n-1) : \Big(\phi(w,\bar{w})\Big)^{n-2} : \right\} \\
&+ \frac{1}{2(z-w)}\sum_{n=1}^{\infty}\left\{\frac{(ia)^n}{n!}2n : \partial\phi(z)\Big(\phi(w,\bar{w})\Big)^{n-1} : \right\} \\
&= -\frac{(ia)^2}{8\pi g(z-w)^2}\sum_{n=2}^{\infty}\left\{\frac{(ia)^{n-2}}{(n-2)!} : \Big(\phi(w,\bar{w})\Big)^{n-2} : \right\} \\
&+ \frac{ia}{(z-w)}\sum_{n=1}^{\infty}\left\{\frac{(ia)^{n-1}}{(n-1)!} : \partial\phi(z)\Big(\phi(w,\bar{w})\Big)^{n-1} : \right\} \\
&= \frac{a^2}{8\pi g}\frac{\mathcal{V}_\alpha(w,\bar{w})}{(z-w)^2} + \frac{ia}{(z-w)}\sum_{n=1}^{\infty}\left\{\frac{(ia)^{n-1}}{(n-1)!} : \partial\phi(z)\Big(\phi(w,\bar{w})\Big)^{n-1} : \right\} \\
&= \frac{a^2}{8\pi g}\frac{\mathcal{V}_\alpha(w,\bar{w})}{(z-w)^2} + \frac{ia}{(z-w)}\sum_{n=1}^{\infty}\left\{\frac{(ia)^{n-1}}{(n-1)!} : \partial\phi(w)\Big(\phi(w,\bar{w})\Big)^{n-1} : \right\} \\
&= \frac{a^2}{8\pi g}\frac{\mathcal{V}_\alpha(w,\bar{w})}{(z-w)^2} + \frac{ia}{(z-w)} : \partial\phi(w)e^{ia\phi(w,\bar{w})} : \\
&= \frac{a^2}{8\pi g}\frac{\mathcal{V}_\alpha(w,\bar{w})}{(z-w)^2} + \frac{\partial_w\mathcal{V}_\alpha(w,\bar{w})}{(z-w)},
\end{aligned} \tag{43}$$

where we replaced $\partial\phi(z)$ with $\partial\phi(w)$ and also we used that

$$\begin{aligned}
\partial_w\mathcal{V}_\alpha(w,\bar{w}) &=: \partial_w e^{ia\phi(w,\bar{w})} : \\
&= ia : \partial_w\phi(w,\bar{w})e^{ia\phi(w,\bar{w})} : .
\end{aligned} \tag{44}$$

III. The free fermion

1. *Verify the OPE $T(z)T(w)$ Eq. (4.66).*

We want to show that:

$$T(z)T(w) = \frac{1/4}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}. \tag{45}$$

We are going to use that:

$$\langle\psi(z)\psi(w)\rangle = \frac{1}{2\pi g}\frac{1}{z-w} \tag{46}$$

$$\langle\partial_z\psi(z)\psi(w)\rangle = -\frac{1}{2\pi g}\frac{1}{(z-w)^2} \tag{47}$$

$$\langle\partial_z\psi(z)\partial_w\psi(w)\rangle = -\frac{1}{\pi g}\frac{1}{(z-w)^3}, \tag{48}$$

and also

$$T(z) = -\pi g : \psi(z)\partial_z\psi(z) : . \tag{49}$$

So,

$$\begin{aligned}
T(z)T(w) &= \pi^2 g^2 : \psi(z)\partial_z\psi(z) :: \psi(w)\partial_w\psi(w) : \\
&= \pi^2 g^2 \Big\{ \underbrace{: \psi(z)\partial_z\psi(z) :: \psi(w)\partial_w\psi(w) :}_{\text{commute once gives a } -} + \underbrace{: \psi(z)\partial_z\psi(z) :: \psi(w)\partial_w\psi(w) :}_{\text{commute twice no sign change}} \\
&+ : \psi(z)\partial_z\psi(z) :: \psi(w)\partial_w\psi(w) : + \underbrace{: \psi(z)\partial_z\psi(z) :: \psi(w)\partial_w\psi(w) :}_{\text{commute once } -} \\
&+ \underbrace{: \psi(z)\partial_z\psi(z) :: \psi(w)\partial_w\psi(w) :}_{\text{commute once gives a } -} + : \psi(z)\partial_z\psi(z) :: \psi(w)\partial_w\psi(w) : \Big\} \\
&= \pi^2 g^2 \Big\{ -\psi(z)\psi(w) : \partial_z\psi(z)\partial_w\psi(w) : + \psi(z)\partial_w\psi(w) : \partial_w\psi(w)\psi(z) : \\
&+ \partial_z\psi(z)\psi(w) : \psi(z)\partial_w\psi(w) : - \partial_z\psi(z)\partial_w\psi(w) : \psi(z)\psi(w) : \\
&- \psi(z)\psi(w)\partial_z\psi(z)\partial_w\psi(w) + \partial_z\psi(z)\psi(w)\psi(z)\partial\psi(w) \Big\} \\
&= \pi^2 g^2 \Big\{ -\frac{1}{2\pi g}\frac{1}{z-w} : \partial_z\psi(z)\partial_w\psi(w) : - \partial_w\psi(w)\psi(z) : \partial_w\psi(w)\psi(z) : \\
&- \frac{1}{2\pi g}\frac{1}{(z-w)^2} : \psi(z)\partial_w\psi(w) : + \frac{1}{\pi g}\frac{1}{(z-w)^3} : \psi(z)\psi(w) : \\
&+ \frac{1}{2\pi^2 g^2}\frac{1}{(z-w)^4} - \frac{1}{4\pi^2 g^2}\frac{1}{(z-w)^4} \Big\} \\
&= \frac{1/4}{(z-w)^4} + \pi^2 g^2 \Big\{ -\frac{1}{2\pi g}\frac{1}{z-w} : \partial_z\psi(z)\partial_w\psi(w) : \\
&+ \frac{1}{2\pi g}\frac{1}{(z-w)^2} : \partial_z\psi(z)\psi(w) : - \frac{1}{2\pi g}\frac{1}{(z-w)^2} : \psi(z)\partial_w\psi(w) : \\
&+ \frac{1}{\pi g}\frac{1}{(z-w)^3} : \psi(z)\psi(w) : \Big\}.
\end{aligned} \tag{50}$$

We are going to Taylor expand now the fields so that

$$\psi(z) = \psi(w) + (z-w)\partial_w\psi(w) \tag{51}$$

$$\partial_z\psi(z) = \partial(w)\psi(w). \tag{52}$$

Do not forget we are dealing with Grassmann numbers. So equation (50) becomes after straightforward calculations:

$$\begin{aligned} T(z)T(w) &= \frac{1/4}{(z-w)^4} + \pi^2 g^2 \Big\{ -\frac{2}{\pi g}\frac{1}{(z-w)^2} : \psi(w)\partial_w\psi(w) : \\ &\quad - \frac{1}{\pi g}\frac{1}{(z-w)} : \partial_w\psi(w)\partial_w\psi(w) : \Big\} \\ &= \frac{1/4}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)}, \end{aligned} \tag{53}$$

where we used that

$$-2\pi g : \psi(w)\partial_w\psi(w) := 2T(w) \tag{54}$$
$$\partial_w T(w) = -\pi g : \partial_w\psi(w)\partial_w\psi(w) : . \tag{55}$$

2. *Verify Eq. (4.36).*

We want to show that

$$\begin{aligned} \langle\psi(z)\psi(w)\rangle &= \sum_{k,q\in\mathbb{Z}} z^{-k-1/2}\, w^{-q-1/2}\langle b_k b_q\rangle \\ &= \frac{1}{2}\frac{\sqrt{\frac{w}{z}} + \sqrt{\frac{z}{w}}}{(z-w)}. \end{aligned} \tag{56}$$

We know that:

$$\psi(z) = \sum_{k\in\mathbb{Z}} b_k z^{-k-1/2}, \qquad \psi(w) = \sum_{q\in\mathbb{Z}} b_q w^{-q-1/2}. \tag{57}$$

And also that

$$\{b_k, b_q\} = \delta_{k+q,0}, \tag{58}$$
$$b_k|0\rangle = 0, \ \ k > 0. \tag{59}$$

- For $k,\, q = 0$ we have:

$$\{b_0, b_0\} = \delta_{0,0} \Rightarrow {b_0}^2 + {b_0}^2 = \delta_{0,0} \Rightarrow {b_0}^2 = \frac{1}{2}. \tag{60}$$

- For $q = -k$ we have:

$$\begin{aligned} \{b_k,\, b_q\} = \delta_{k+q,0} &\Rightarrow b_k\, b_{-k} + b_{-k}\, b_k = 1 \\ &\Rightarrow b_k\, b_{-k} = 1 - b_{-k}\, b_k. \end{aligned} \tag{61}$$

Then,

$$\begin{aligned} \langle\psi(z)\psi(w)\rangle &= \sum_{k,\in\mathbb{Z}}\sum_{q,\in\mathbb{Z}} z^{-k-1/2}\, w^{-q-1/2}\langle b_k b_q\rangle \\ &= z^{-1/2}w^{-1/2}\langle b_0\, b_0\rangle + \sum_{k,\in\mathbb{Z}^*}\sum_{q,\in\mathbb{Z}^*} z^{-k-1/2}\, w^{-q-1/2}\langle b_k b_q\rangle \\ &= \frac{1}{2\sqrt{zw}}\langle 0|0\rangle + \frac{1}{\sqrt{zw}}\sum_{k,\in\mathbb{Z}^*}\sum_{q,\in\mathbb{Z}^*} z^{-k}\, w^{-q}\langle b_k b_q\rangle, \end{aligned} \tag{62}$$

where we used equation (60). Then we can use the anti-commutator for the b_k, b_q and see that this is non-zero only for $q = -k$. Also from equation (59) we have that $q < 0$ which leads to $k > 0$. Hence we have:

$$\begin{aligned}
\langle \psi(z)\psi(w)\rangle &= \frac{1}{2\sqrt{zw}}\langle 0|0\rangle + \frac{1}{\sqrt{zw}}\sum_{k=1}^{\infty} z^{-k}\,w^{k} \\
&= \frac{1}{\sqrt{zw}}\left(\frac{1}{2} + \sum_{k=1}^{\infty}\left(\frac{w}{z}\right)^{k}\right) \\
&= \frac{1}{\sqrt{zw}}\left(\frac{1}{2} + \sum_{k+1=1}^{\infty}\left(\frac{w}{z}\right)^{k+1}\right) \\
&= \frac{1}{\sqrt{zw}}\left(\frac{1}{2} + \frac{w}{z}\sum_{k=0}^{\infty}\left(\frac{w}{z}\right)^{k}\right) \\
&= \frac{1}{\sqrt{zw}}\left(\frac{1}{2} + \frac{\frac{w}{x}}{1-\frac{w}{z}}\right) \\
&= \frac{1}{\sqrt{zw}}\left(\frac{1}{2} + \frac{w}{z-w}\right) \\
&= \frac{1}{2\sqrt{zw}}\left(\frac{w+z}{z-w}\right) \\
&= \frac{1}{2}\frac{\frac{w}{\sqrt{zw}} + \frac{z}{\sqrt{zw}}}{(z-w)} \\
&= \frac{1}{2}\frac{\sqrt{\frac{w}{z}} + \sqrt{\frac{z}{w}}}{(z-w)},
\end{aligned} \tag{63}$$

where in the fourth line we used that

$$\sum_{k=0}^{\infty} x^{k} = \frac{1}{1-x}, \qquad x < 1. \tag{64}$$